SHORT ANSWER TYPE QUESTIONS ON MATHEMATICS & STATISTICS

Q1. If $a_1 < a_2 < \ldots\ldots\ldots < a_m$, $b_1 < b_2 < \ldots\ldots <$, b_n and also $\sum_{i=1}^{m} |ai - x| = \sum_{j=1}^{n} |bj - x|$, where x is any real number then prove that ai = bj for all i and n = m.

Solution: let f(x)=$|a_1$-x|+$|a_2$-x|+…..+$|a_m$-x|

And g(x) = $|b_1$-x|+$|b_2$-x|+…..+$|b_n$-x|.

Then we know only points of non-differentiability of f(x) is $a_1, a_2, \ldots\ldots\ldots, a_m$,and only points of non-differentiability of g(x) is $b_1, b_2, \ldots\ldots\ldots, b_n$,

Since, m, n are finite numbers and also given that f(x)=g(x)

So, we may write, $\frac{f(ai+h)-f(ai)}{h} = \frac{g(ai+h)-g(ai)}{h}$ $\forall$ h.

So, RHL{f'(ai)}=RHL {g'(ai)}

And also, LHL{f'(ai)}=LHL {g'(ai)}

But as f(x) is non-differentiable at x=ai,

So, LHL{f'(ai)}≠RHL {f'(ai)}, LHL{g'(ai)}≠RHL {g'(ai)→g(x) is also not differentiable at x=ai.

Now, since both the functions are equal so the points of discontinuity are same so m=n.

To show the another part, we need to show ai=bi.

In a similar way, we can say, for any given b_r there exists a_p

Such that $b_r = a_p$.

So, $\{a_1, a_2, \ldots\ldots\ldots, a_m\}$ and $\{b_1, b_2, \ldots\ldots\ldots, b_n\}$ has one –to-one and onto correspondence.

Therefore, m=n and every $a_i = b_j$ if i = j.

Q2. Suppose w_1 and w_2 are subspaces of Ψ^4 spanned by {(1, 2, 3, 4) , (2, 1, 1, 2)} and {(1, 0, 1, 0), (3, 0, 1, 0)} respectively. Find a basic of $w_1 \cap w_2$. Also find a basis of $w_1 + w_2$ containing {(1, 0, 1, 0), (3, 0, 1, 0)}. (Ψ : The set of all real numbers)

Solution: w_1 ={(1, 2, 3, 4) , (2, 1, 1, 2)} $w_2 = \{(1, 0, 1, 0), (3, 0, 1, 0)\}$

Now we will calculate dim($w_{1\upsilon} w_2$) which is equal to number of independent rows in

1	2	3	4
2	1	1	2
1	0	1	0

i.e. Rank(A)=4.

Now, dim($w_{1\cup} w_2$)=dimw_1+dimw_2-dim($w_1 \cap w_2$)

⇨ 4 =2 + 2 - dim ($w_1 \cap w_2$)

⇨ dim ($w_1 \cap w_2$)=0.

i.e. basis of ($w_1 \cap w_2$)={(0, 0, 0, 0)}

⇨$\Psi^4 = w_{1\oplus} w_2$

⇨ basis of w_2 can be extended to form basis of $w_1 + w_2$ which is given by

= {(1, 0, 1, 0), (3, 0, 1, 0), (0, 1, 0, 0), (0, 0, 0, 1)}

Q3. Two players p_1 and p_2 are playing the final of a chess championship, which consist of a series of matches. Probability of p_1 winning a match is $^2/_3$and for p_2 is $^1/_3$. The winner will be one who is ahead by 2 games as compared to the other player and wins at least 6 games. Now, if the player p_2 wins first four matches, find the probability of p_1 winning the championship.

Solution:- p_1 can win in the following mutually exclusive ways:

(a) p_1 wins the next six matches.
(b) p_1 wins five out of next six matches, so that after next six matches score of p_1 and p_2 are tied up. This is continued up to '2_n' matches (n≥ 0) and finally p_1 wins 2 consecutive matches.

Now, probability of case (a) =$(\frac{2}{3})^6$ and probability of tie after 6 matches (in case (b))=
=$(\frac{2}{3})^5(\frac{1}{3})$ =$\frac{2^6}{3^5}$.

Now probability that scores are still tied up after another next two matches=$\frac{2}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{2}{3}=\frac{4}{9}$.

[1st match is own by p_1 and 2nd by $p_{2,}$or , by reversively]

Similarly probability that scores are still tied up after another '2n' matches=$(\frac{4}{9})^n$.

⇨Total probability of p_1 winning the championship

$= (\frac{2}{3})^6 + \frac{2^6}{3^5} (\sum_{n=0}^{\infty} (\frac{4}{9})^2 \ (\frac{2}{3})^2)$

$= (\frac{2}{3})^6 + \frac{2^6}{3^5} (\frac{2}{3})^2 (\frac{1}{1-\frac{4}{9}})$

$= \frac{17}{5} (\frac{2}{3})^6$

$= \frac{1088}{3645}$.

Q4. Let $X_1, X_2, \dots, X_n$ be a random sample drawn from a continuous distribution. The random variables are ranked in the increasing order of magnitude. R_i be the rank of the ith sample. Find the correlation coefficient between R_1 and R_2.

Solution:- R_i be the rank of X_i.

R_i be the random variable such that P$(R_i = r_i) = \frac{1}{n}$; ri=1(1)n .

$\therefore \sum_{i=1}^{n} R_i = \frac{n(n+1)}{2}$, a constant quantity.

And since $R_1, R_2, \dots, R_n$ is identical random variable, now R_i is th random variable and $\sum_{i=1}^{n} R_i$ is a constant.

$\therefore$ cov$(R_1, \sum_{i=1}^{n} R_i)$=0

⇨ cov$(R_1, R_1 + R_2 + \cdots + R_n,)$=0

⇨var(R_1) +cov(R_1, R_2) + ….+cov(R_1, R_n)=0

⇨var(R_1)+(n-1). Cov(R_1, R_2)=0 [$\because R_i$'s are identically distributed; cov(R_i, R_j)= cov(R_i)]

⇨cov(R_1, R_2)=-$\frac{var(R_i)}{(n-1)}$

$= -\frac{\frac{n^2-1}{12}}{(n-1)} = -\frac{(n+1)}{12}$

$\therefore \boldsymbol{\rho} = \frac{cov(R_1,R_2)}{\sqrt{var(R_1)var(R_2)}} = \frac{-\frac{n+1}{12}}{\frac{(n+1)(n-1)}{12}} = -\frac{1}{(n+1)}$.

Q5. Let X and Y be two random variables with joint P. D. F. f(x, y) = 1 if –y< x<y , 0<y <1

0 elsewhere

Find the regression equation of Y on X and that of probability density function.

Solution:-

Here –y< x<y and 0<y <1

⇨ -1< x <1 , which is the marginal range of x.

Again, y> -x and y > x

∴ y > max (x_1 -x)

∴ max (x_1 -x) < y < 1

∴ Marginal PDF of X is given by,

$f_X(x) = \int_{\max(x_1 -x)}^{1} (x, y)\, dy$, -1 < x < 1

Case –I:- -1 < x <0

∴ max (x_1 -x) = -x

∴ $f_X(x) = \int_{-x}^{1} dy$, if -1 < x < 0

= 1+ x if -1 < x <0

Case –II:- 0 < x< 1

Max(x, -x) = x,

∴ $f_X(x) = \int_{x}^{1} dy$, if 0 < x < 1

= 1- x if 0< x <1

Marginal PDF of Y is given by.

$f_y(y) = \int_{-y}^{1} dx$, if 0 <y < 1

= 2y if 0 <y < 1

Case 1. -1 <x < 0,

The conditional distribution of Y given X= x is given by,

$f_{y/x}(y) = \frac{f(x,y)}{f_x(x)} = \frac{1}{1+x}$ if –x < y < 1

∴ E (Y|X) = $\int_{-x}^{1} \frac{y\,dy}{(1+x)} = \frac{1}{(1+x)} . \frac{1}{2}(1-x^2) = \frac{1-x}{2}$.

Case 2. 0 < x < 1,

The conditional distribution of Y given X=X is given by,

$f_{Y|X}(y) = \frac{f(x,y)}{f(x)} = \frac{1}{1-x}$ if x < y < 1

Similarly, E (Y|X) = $\frac{1+x}{2}$

∴ If -1 < x < 1, then regression equation of Y on X is given by,

Y= $\frac{1-|x|}{2}$.

The conditional distribution of X given Y = y is given by,

$f_{X|Y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{2y}$ if –y < x < y

∴ E (X|Y)=0

∴ Regression equation of X and Y is given by x =0.

Q6. (a) Let $f_n$be a sequence of continuous real valued functions on, [0,1] which converges uniformly to f . Prove that $\lim_{n\to\infty} f_n(x_n) = f\left(\frac{1}{2}\right)$for any sequence $\{x_n\}$converges to ½.

(b) Must the conclusion still hold if the convergence is only point wise? Explain.

Solution:- (a) Let $\{x_n\}$ be a sequence in [0,1] with $x_{n\to 1/2}$ as n→ Fix ∊>0 and let $N_0 \in N$ be such that n ≥ N_0 implies $|f_n(x) - f(x)|$< for all x∊[0,1]. Let δ>0 be such that $|f(x) - f(y)|$∊/2 ∀ x, y ∊ [0,1] with $|x-y| < \delta$. Finally, let $N_1 \in N$ be such that n ≥ N_1 $implies$ $|x_n - 1/2|$<δ. Then n ≥ Max { N_0, N_1} inplise

$|f_n(x_n) - f(\frac{1}{2})| \leq |f_n(x_n) - f(x_n)|$ + $|f(x_n) - f(\frac{1}{2})|$

$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

(b) Suppose the convergence is only pointwise.

Then the conclusion is false, demonstrating by an counter example:

Defining $f_n(x)$ to be the function,f(x)={ 0 , if $0 \leq x < \frac{1}{2} - \frac{1}{2n}$

2nx-(n-1), if $\frac{1}{2} - \frac{1}{2n} \leq x < \frac{1}{2}$

1 , if ½

i.e. $f_n(x)$is constantly zero for, x< ½ - $\frac{1}{2n}$, then it increases linearly until it reaches '1' at x= ½ , and then it remains constantly '1' for x> ½ .

Now, define the sequence, $x_n = \frac{1}{2} - \frac{1}{n}$,

Then f(x_n)=0 $\forall\, n \in N$ and $x_{n \to \frac{1}{2}}$ for n→∞.

Therefore, f(½)=1≠0 =$\lim_{n\to\infty} f_n(x_n)$.

Q7. Let $\{x_n : n \geq 0\}$ be a sequence of real numbers such that $x_{n+1} = \lambda x_n + (1 - \lambda)x_{n-1}, n \geq 1$, for some 0< $\lambda < 1$.

(a) show that $x_n = x_0 + (x_1 - x_0)\sum_{k=0}^{n=1}(\lambda - 1)^k$.

(b) Hence, or, otherwise, show that x_n converges and find the limit.

<u>Solution</u> :- $x_{n+1} - x_n = \lambda x_n + (1-\lambda)x_{n-1} - \lambda x_{n-1} + \lambda x_{n-1} - x_n$

$=(\lambda - 1)x_n + x_{n-1}(1 - \lambda - \lambda + \lambda)$

$= (\lambda - 1)[x_n - x_{n-1}]$

$= (\lambda - 1)^2[x_{n-1} - x_{n-2}]$

⋮

$= (\lambda - 1)^n(x_{1-}x_0)$

$\therefore x_n - x_{n-1} = (\lambda - 1)^{n-1}(x_{1-}x_0)$

$x_{n-1} - x_{n-2} = (\lambda - 1)^{n-2}(x_{1-}x_0)$

⋮

$x_{1-}x_0 = (\lambda - 1)^0(x_{1-}x_0)$

Adding we get, $x_{n-}x_0 = (x_{1-}x_0)\sum_{k=0}^{n=1}(\lambda - 1)^k$

$\therefore x_n = x_0 + (x_{1-}x_0)\ \sum_{k=0}^{n=1}(\lambda - 1)^k$

$\therefore$ Lt $_{n\to\infty}$ $x_n = x_0 + (x_{1-}x_0).\frac{1}{1-\lambda+1}$, as n→∞.

$=x_0 + (x_{1-}x_0).\frac{1}{2-\lambda}$.

Q8. Let f:ℝ→ℝ be a continuous function with |f(x)-f(y)|≥ $|x - y$ for every x,y,∈ℝ. Is f one –to-one? Show that there can't exist three points a, b, c∈ℝ with a< b< c such that f(a)<f(c)< f(b).

Solution:- f: ℝ→ℝ such that |f(x)-f(y)|≥ $|x - y|$

or, $|\frac{f(x)-f9y)}{x-y}|\geq 0\ \forall\ x, y \in$ ℝ.

⇨$|\frac{f(y+h)-f(y)}{h} \geq 0$, taning x=y+h.|

⇨|f'(y)|≥ 0

⇨f is either an increasing or decreasing function.

⇨$\forall\ a < b < c \to f(a) < f(b) < f(c)$

Or, a> b> c⟶ $f(a) < f(b) < f(c)$

i.e. a< b<c   s.t. $f(a) < f(b) < f(c)$.

Q9. (a) Let ṵ and ṽ eigenvectors of A corresponding to the eigenvalues 1 and 3, respectively. Prove that ṵ +ṽ is not an eigenvector of A.

(b) Let A and B be real matrices such that the sum of each row of A is 1 and the sum of each row of B is 2. Then show that 2 is an eigenvalue of AB.

Solution:-As ṵ and ṽare given eigen vectors corresponding to eigen values 1 and 3, so,

$A_u = 1.u\ ; A_v = 3.v = 3v.$

⇨ A(u+v)=u+3v.

As, RHS is not multiple of u+v, so, u+v can't be eigen vector of A.

(b) A=$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$, B=$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

⇨AB=$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

|AB -λI|=$(1-\lambda)^2-1$=0

⇨$1-\lambda$=$\pm$ 1

⇨ $\lambda = 0, 2.$

So, 2 is an eigen value of AB.

Q10. Let A and B be n×n real matrices such that $A^2 = A, B^2 = B$. Let I-(A+B) is invertible. Show that R(A)=R(B).

Solution:- A[I –(A+B)]=A -A^2-AB

= $A^2 - A^2$-AB

= -AB.

And,[I –(A+B)]B=B -AB - B^2

=B^2- AB - B^2

=-AB

∴ rank(A)= rank[A(I – A-B)]=rank(-AB)=rank(b).

Q11. Let P be a matrix of order n>1 and entries are positive integers. Suppose P^{-1} exists and has integer entries, then what are the set of possible values of |p| ?

Solution:- P has integer entries,

⇨$\lambda_1 + \lambda_2 + \cdots . + \lambda_n$=trace(P)=integer.

⇨ $\sum_{i<j} \lambda i \, \lambda j$ = sum of minors

⋮

⇨$\prod_{i=1}^{n} \lambda i$|p| = integer,

Then the eigen-values of P^{-1} are $\frac{1}{\lambda i}$ and they are also integers

⇨ $\lambda i = \frac{1}{\lambda i}$

⇨ $\lambda i = \pm 1$

So, |P|=$\prod_{i=1}^{n} \lambda i = \pm 1.$

Q12. Let X, Y be a bivariate normal vector such that E(X)=E(Y)=0 and V(X)=V(Y)=1. Let s be a subset of $\mathbb{R}^2$ and defined by S={(a, b) : (ax+ by) is independent of Y}.

(i) show that S be a sub space,

(ii) Find its dimension.

Solution:- S={(a, b) : (ax+by) is independent of Y }

(i) (a_1, b_1), (a_2, b_2)∈S.

Then $a_1x + b_1y$ is independent of y, similarly,

$a_2x + b_2y$ is independent of y.

⇨ (**α**a_1+**β**a_2)x +(**α**b_1 + $\boldsymbol{\beta b_2}$)y is independent of y.

⇨ (**α**a_1+**β**a_2, **α**b_1 + $\boldsymbol{\beta b_2}$ ∈**s).**

⇨ (a_1, b_1) +**β**(a_2, b_2)∈ s ∀ $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ ∈ ℝ

Hence, S is a subspace.

(ii) (a, b)∈S.

⇨ax +by is independent of y.

⇨ cov(ax +by, y) =0

⇨ acov(x,y) + bcov(y, y) =0

⇨a**ρ** +b=0, since, cov(x, y)=**ρ**, cov(y,y)=var(y)=1 as,E(x)=E(y)=0 &v(x)=v(y)=1'

⇨ b= -a**ρ**.

∴ (a, b)=a(1-**ρ**), a∈ℝ.

∴ S={(a, b) : (a,b)=a(1-**ρ**); a∈ℝ }

∴ dim(s)=1.

Q13. In a knockout tournament, $\boldsymbol{2^n}$ equally skilled players namely, $\boldsymbol{s_1, s_2, s_3, \ldots\ldots, s_{2n}}$ are participating. In each round, player are divided in pairs at random and winner from each pair moves in the next round. If $\boldsymbol{s_2}$ reaches semi-final , then find the probability that $\boldsymbol{s_1}$ will win the tournament.

Solution:- In a knockout tournament, 2^n equally skilled players namely, $s_1, s_2, s_3, \ldots\ldots, s_{2n}$ are participating.

Let E_1 be the event that s_1 wins the tournament and

E_2 be the event that s_2 reaches the semifinal.

We are to obtain P(E_1/E_2).

Since all the players are of equal skill and there will be four person in the semifinal.

So, P(E_2) =$\frac{2^n-1\ c_3}{2^n\ c_4}$=$\frac{4}{2^n}$.

P($E_1 \cap E_2$) =probability that s_1 and s_2 both are in the semifinal & then s_1 wins the semifinal and also in final

=$\frac{2^n-2c_2}{2^n\ c^4} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{2^n(2^n-1)}$

Hence, P[E_1/E_2]=$\frac{P(E_1 \cap E_2)}{P(E_2)}$

=$\frac{3\ .2^n}{2^n(2^n-1).4}$=$\frac{3}{4(2^n-1)}$.

Q14. Let Y, $\boldsymbol{Y_2, Y_3}$ be i.i.d. continuous r.v.s for i=1, 2. Define $\boldsymbol{U_i}$ as $\boldsymbol{U_i}$ =1 if $\boldsymbol{Y_{i+1} > Y_i}$

=0 ow

Find the mean and variance of $\boldsymbol{U_1 + U_2}$.

Solution:- E(U_i)=1.P[$Y_{i+1} > Y_i$]=$\frac{1}{2}$

E $(U_i)^2$=1^2.P[$Y_{i+1} > Y_i$]=$\frac{1}{2}$

V (U_i) =$\frac{1}{2} - \frac{1}{4}$ =$\frac{1}{4}$

E $(U_1 + U_2)$= $\frac{1}{2} + \frac{1}{2}$ =1.

E (U_1U_2) =1.1.P[$Y_2 > Y_1, Y_3 > Y_2$] = P[Y_3>$Y_2 > Y_1$]=$\frac{1}{6}$.

Cov(U_1, U_2)= E(U_1U_2)-E(U_1)E(U_2) = -$\frac{1}{12}$

∴ V($U_1 + U_2$)=V(U_1)+V(U_2)+2cov(U_1, U_2) =$\frac{1}{3}$.

Q15. A and B have respectively (n+1) and n coins. If they toss their cion simultaneously. What is the probability that, ____

i>A will have more heads than B.

ii> A and B will have an equal number of heads.

iii> B will have more heads than A.

Soln:- Let us define the random variable as follows,

X= no. of heads obtained by A.

Y= No. of heads obtained by B.

X ~bin (n+1, $\frac{1}{2}$)

Y ~ bin (n, $\frac{1}{2}$)

Then, (n+1-X) ~ bin (n+1, $\frac{1}{2}$)

(n-Y) ~ bin (n, $\frac{1}{2}$)

i> P(A will have more heads than b)
=(x>Y)
=P(n+1-X > n-Y)
=P (Y >X -1)
=P(Y≥ X)
=1 –P(X>Y)

∴ 2P(X>Y)=1

⇨ P(X>Y)=$\frac{1}{2}$.

ii> P(A and b have equal number of heads)
=P(X=Y)
$= \sum_{i=1}^{n} P(X = i0\ P(Y = i)$
$= \sum_{i=1}^{n} \binom{n+1}{i} \left(\frac{1}{2}\right)^{i} \left(\frac{1}{2}\right)^{n+1-i} .\binom{n}{i} \left(\frac{1}{2}\right)^{i} \left(\frac{1}{2}\right)^{n-i}$
$=\sum_{i=1}^{n} \binom{n+1}{i} . \binom{n}{i} \left(\frac{1}{2}\right)^{2n+1} = \left(\frac{1}{2}\right)^{2n+1} \sum_{i=1}^{n} \binom{n+1}{i} \binom{n}{i}$
$= \left(\frac{1}{2}\right)^{2n+1} \sum_{i=1}^{n} \frac{(n+1)!}{i!(n-i+1)!} . \frac{n!}{i!(n-i)!}$
$= \left(\frac{1}{2}\right)^{2n+1} \binom{2n+1}{n}$

iii> P(B have more heads than A)
=P(y>x)
=1-P(X≥ Y)
= 1-P(X=Y)-P(X>Y)

$=1-\left(\frac{1}{2}\right)^{2n+1}\binom{2n+1}{n}-\frac{1}{2}$

$=\frac{1}{2}[1-\left(\frac{1}{2}\right)^{2n}\binom{2n+1}{n}]$

Q16. A book of N pages contains on the average λ misprints per page. Estimate the probability that a page drawn at random contains,

(a) at least one misprints.

(b) More than k misprints.

Solution:- Let us define the random variable X as follows,

X= no. of misprints per page,

The book contains λ misprints per page on an average.

Since the number of trials i.e. the no. of words is very large and probability of a misprint is very small, hence according to the definition of poisson distribution,

X ~ p(λ)

$\therefore$ P(X = x) = $\frac{e^{-\lambda}.\lambda^{x}}{x1}$, x=0, 1, 2, …; λ>0

=0 ,ow

(a)P(at least one misprint)

= P(X≥ 1)

= 1-P(X<1)

= 1- P(X=0)

= (1- $e^{-\lambda}$)

(b) P(more than k misprints)

=P(x>k)

=P(X≥ k-1)

= 1 – P(X≤ k-1)

= 1- $\sum_{x=0}^{k=1}\frac{e^{-\lambda}.\lambda^{x}}{x!}$

Q17. A certain mathematician carries two match boxes in his pocket, each time he wants to use a match, he selects one of boxes at random. Each pocket contain n matchsticks.

(a) Find the distribution of the number of sticks in one box, while the other is found empty.

(b) Also find the distribution of the number of sticks remaining in one box become empty.

Solution:- (a) Let us define a r.v. X denoting the number of the matchsticks remaining in the match box cohen the other box is found empty.

Let X_{ij} , I, j, i≠j denotes the number of matchsticks remaining in the ith box cohen the jth box is found to be empty.

The mass points of X are 0, 1, …., N

For any such mass point x,

P[X=x] =P(X_{12}=x) +P(X_{21}=x)

We consider the distribution of X_{12}.

The second box will be found empty if the box is chosen for the (N+1) th time. At that time the first box contain x matches if (N-x) matches have already taken from it. If the selection of the second box is regarded as success, then the event.

P[X_{12}=x]=P[(N-x) failures occur preceeding the (N+1)th success].

=p[Z=N-x], where Z ~ NB(N+1,$\frac{1}{2}$)

=(n+1+N-x-1)$(\frac{1}{2})^{N-x}$ $(\frac{1}{2})^{N+1}$

N-x

=(2N-x) $(\frac{1}{2})^{2N-x+1}$

N-x

Similarly, P[X_{21}=x] =(2N-x) $(\frac{1}{2})^{2N-x+1}$

N-x

∴ P[X=x] =(2N-x) $(\frac{1}{2})^{2N-x}$, x= 0, 1, 2, …, n.

N-x

(b) Let us define a random variable Y denoting the number of matchsticks remaining in a matchbox when the other match box becomes empty.

Let Y_{ij} , I, j, i$\neq j$denotes the number of matchsticks remaining in the ith box when jth box becomes empty.

The mass points of Y are 0, 1, 2, …, n.

P[Y=y]=p[Y_{21}=y]+P[Y_{12}=y]

Now, P[Y_{12}=y]p[Z=n-y] , Z ~ N.B(N, $\frac{1}{2}$)

=($^{N+N-y-1}C_{N-y}$) $(\frac{1}{2})^{N}(\frac{1}{2})^{N-y}$

=($^{2N-y-1}C_{N-y}$) $(\frac{1}{2})^{2N-y}$

Similarly, P[Y_{21}=y]=($^{2N-y-1}C_{N-y}$) $(\frac{1}{2})^{2N-y}$

∴ P[Y=y]]=($^{2N-y-1}C_{N-y}$) $(\frac{1}{2})^{2N-y+1}$

Q18. A drunk man performed a random walk over the position 0, $\pm 1, \pm 2$,…. The drunk man stars from the point o. He takes successive unit steps with probability p at right and probability (1-p) at left. His steps are independent. X be a location of the drunk man after taking n- steps,

Find the distribution of $\frac{(n+X)}{2}$ and find out E(X).

Solution:- R denotes no. of steps at right after taking n steps.

∴ R~ Bin(n, p)

L denotes no. of steps at left after taking n steps.

∴ L~ Bin(n,1-p)

Let us define,

X: the position of the drunk and after n steps.

R + l= n,

r- l=X.

∴ 2R= n+X ⇨ R=$\frac{n+X}{2}$ ~ Bin(n,p)

∴ E($\frac{n+X}{2}$)=nP

⇨ E(X) =2[np - $\frac{n}{2}$]

=2n (p - $\frac{1}{2}$)

=n (2p-1).

Q19. Let X be an R.V. with mean μ and variance σ^2>0.

If ξ_q denotes the q^{th} quantile of X, show that

$$\mu-\sigma\sqrt{\frac{1-q}{q}} \leq \xi_q \leq \mu+\sigma\sqrt{\frac{q}{1-q}}.$$

ANS:- We know that ξ_q statistics the incauality p(X≤ ξ_q)≥q

∴ P($\frac{X-\mu}{\sigma} \leq \frac{\xi_q-\mu}{\sigma}$) $\geq q$

If ξ_q <μ, i.e. $\frac{\xi_q-\mu}{\sigma}$<0, we have from one sided chebyshev's inequality,

$$q\leq P[\frac{X-\mu}{\sigma} \leq \frac{\xi_q-\mu}{\sigma}] \leq \frac{1}{1+(\frac{\xi_q-\mu}{\sigma})^2}$$

$$\therefore q\leq \frac{1}{1+(\frac{\xi_q-\mu}{\sigma})^2}$$

$$⇨ (\frac{\xi_q-\mu}{\sigma})^2 \leq \frac{1-q}{q}$$

$$⇨ -\sqrt{\frac{1-q}{q}} \leq \frac{\xi_q-\mu}{\sigma} \leq \sqrt{\frac{1-q}{q}}$$

⇨ $\mu-\sigma\sqrt{\frac{1-q}{q}} \leq \xi_q \leq \mu+\sigma\sqrt{\frac{1-q}{q}}$.(Proved)

Q20. Let g be a non – negative non decreasing function, prove that if E(g(|X − μ|)) exists, where μ= E(X), then prove that if P[|X − μ| > t]< $\frac{(g(|X-\mu|))}{g(t)}$

ANS:- P[g|X − μ| > $g(t)$]≤ $\frac{E\{g|X-\mu|\}}{g(t)}$

But, g|X − μ| > $g(t)$

$\Leftrightarrow |X - \mu| > t$ [∵ g is non decreasing & non-negative function]

∴ $P[|X - \mu| > t] < \frac{E\{g|X-\mu|\}}{g(t)}$. (Proved)

Q21. For a Laplaces distribution with PDF $f(x) = \frac{1}{2} e^{-|x|}$, $-\infty < x < \infty$.

Find the minimum probability of an observation lying with in the mean $\pm$ 3 s. d. interval.

ANS:- $P(|X - \mu| \leq 3\sigma$

$= P(\mu - 3\sigma \leq X \leq \mu + 3\sigma)$

$= \frac{1}{2}\int_{\mu-3\sigma}^{\mu+3\sigma} e^{-|x|}\, dx$

$= \int_{0}^{\mu+3\sigma} e^{-x} dx$ [Since the integrand is an even function]

$= -e^{-x} \int_{0}^{\mu+3\sigma}$

$= 1 - e^{-(\mu+3\sigma)}$

=.95 [X~ laplace (0, 1)]

By Chebyshev's inequality,

$P[|X - \mu| \geq 3\sigma] \leq \frac{1}{3^2}$

⇨ $P[|X - \mu| \leq 3\sigma] \geq 1 - \frac{1}{9} = \frac{8}{9} = .88$

Hence the given probability and the Chebyshev's upperbound is nearer to each other.

Q22. For the r. v. X having the following PDF

$f(x) = \frac{e^{-x}.x^{\lambda}}{\sqrt{\lambda+1}}$, x> 0 show that $P(0 < X < 2(\lambda + 1)) > \frac{\lambda}{\lambda+1}$

ANS:- $E(X) = (\lambda + 1) = Y(X)$

From Chebyshev's inequality,

$P[|\frac{X-\mu}{\sigma}| < t] > 1 - \frac{1}{t^2}$ $1 - \frac{1}{t^2} = \frac{\lambda}{\lambda+1}$

⇨ $P[-\sigma t < (X - \mu) < \sigma t] > 1 - \frac{1}{t^2}$ ⇨ $t = \sqrt{\lambda + 1}$

⇨ $P[-(\sqrt{\lambda + 1})(\sqrt{\lambda + 1}) < (X - \sqrt{\lambda + 1}) < \lambda + 1] > 1 - \frac{1}{\lambda+1}$

⇨ P [0, X, 2(λ + 1)]> $\frac{\lambda}{\lambda+1}$

Q23. Let the random variables X and Y have the joint probability density function(x, y) given by

f(x, y)= $\boldsymbol{y^2 e - y^{(x+1)}; x \geq 0, y \geq 0}$

= 0 ;otherwise

Are the random variables x and Y independent? Justify your answer .

Solution:- The marginal p d f of x is given by :

F(x) =$\int_0^\infty y^2\, e^{-y(x+1)} dy$ let, $y(x+1)$=t, (x+1) dy =dt.

= $\int_0^\infty \frac{t^2}{(x+1)^2}\, e^{-t}. \frac{1}{(x+1)}\, dt$

= $\frac{1}{(x+1)^3} \int_0^\infty t^2 e^{-t}\, dt$

=$\frac{\sqrt{(3)}}{(x+1)^3}$ =$\frac{2}{(x+1)^3}$; x ≥ 0

The marginal p d f of Y is given by :

F(y) = $\int_0^\infty y^2\, e^{-y(x+1)} dx$

= $\int_0^\infty \frac{y^2}{-y} . e^{-y(x+1)}$

= ye^{-y} ;y≥0

As, f(x, y)≠f(x), f(y)

So, X and Y are not independent.

Q24. Let X and Y are i.i.d. with P[X= x] = $\boldsymbol{\frac{1}{x} - \frac{1}{x+1}}$**, x=1, 2, …**

Find E[Min(X, Y)].

Solution:- Let T= min (x, Y).

P[T= t]= P[X=t, Y>t]+ P[Y=t, X>t] + P[x=t, Y=t]

=P[x=t]P[Y> t]+ P[X> t]P[Y=t]+p [X=t]P[Y= t]

Now, P[Y≤t]= P[Y=1]+ P[Y=2]+…+P[Y=t]

$=(1-\frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{t} - \frac{1}{t+1})$

$= 1- \frac{1}{t+1}$

⇨ $P[Y>t] = \frac{1}{t+1}$

Similarly, $P[X> t] = \frac{1}{t+1}$

Hence, $P[T=t] = (\frac{1}{t} - \frac{1}{t+1}). \frac{1}{t+1} + \frac{1}{t} - \frac{1}{t+1}). \frac{1}{t+1} + (\frac{1}{t} - \frac{1}{t+1})^2$

$= \frac{1}{t(t+1)^2} + \frac{1}{t^2(t+1)}$

$\therefore E(T) = \sum_{t=1}^{\infty} \frac{1}{(t+1)^2} + \sum_{t=1}^{\infty} \frac{1}{t\,(t+1)}$

$=(\frac{\pi^2}{6} - 1) + \sum_{t=1}^{\infty} (\frac{1}{t} - \frac{1}{t+1})$

$= \frac{\pi^2}{6} -1 + 1$

$= \frac{\pi^2}{6}$

Q25. Suppose a random vector (X, Y) has joint probability density function

f(x, y) =3y on the triangle bounded by the lines y=0, y=1 –x and y= 1 +x

find the marginal PDF of X and Y. Compute (Y| X≤ $\frac{1}{2}$).

<u>Solution</u>:- The joint PDF of the random vector

(X, Y) is given by

F(x, y) = 3y, on the shaded triangle of figure 1.

From the figure, the range of the marginal

Distributions of X and Y are given by,

$0 < x < 1$, $-1 < y < 1$, respectively,

Now, $f(x, y) = 3y$ if $1 - x < y < 1 + x$, $0 < x < 1$

Now, $1 + x > y$

⇨ $x > y - 1$

And $1 - x < y$

⇨ $x-1 > -y$

⇨ $x > 1 - y$

∴ $x > \max \{(1-y), (y-1)\}$

∴ $\max \{(1-y), (y-1)\} < x < 1$

Case I:- $-1 < y < 0$

If $-1 < y < 0$

⇨ $-2 < y-1 < -1$

And $1 > -y > 0$

⇨ $2 > 1-y > 1$

∴ $\max \{(y-1), (1-y)\} = 1-y$

∴ $1-y < x < 1$

Q26. Let x be a continuous random variable with distribution function f(x), which is such that F(a+x) +F(a-x) =1 for some fixed a.

i> Show that E(X) =a

ii> If y be an another r. v. defined as

Y=0 if X < a ; 1 if X > a

Then S.T. Y and Z=|X-a| will be independently distributed.

ANS:- i> It is given that, $F(a+x) + F(a-x) = 1$

From the above equation it is clear that the disth of X is symmetric about 'a',

Hence, $E(X-a) = 0$

⇨ $E(X)=a$

Ii > It is given that,

$Y = \{ 0$ if $X < a$; 1 if $X \geq a$

And $Z=|X-a|$

Now form the equation, F(x+a) +F(a-x)=1,

It is clear that F(a)=$\frac{1}{2}$ [since the distribution is symmetric about 'a']

∴ Y= { 0 with prob. $\frac{1}{2}$; 1, with prob. $\frac{1}{2}$

Now, for same Z > 0

P[Z≤z , Y=0]

=P [|X-a|≤ z , X < a]

= P[-x +a≤ z≤ X-a , X < a]

= P[a-z≤ X≤ a+z, X < a]

=P [a-z≤ X≤ min (a+z, a)]

= P [a-z≤ X≤a]

=F(a) –F(a-z)

= $\frac{1}{2}$ – F (a-z)

= $\frac{1}{2}$ [F(a+z) –F(a- z)]

= $\frac{1}{2}$ P [a-z ≤ X ≤ a+ z]

= $\frac{1}{2}$ P [|X|≤ a+z]

= $\frac{1}{2}$ P [|X - a|≤z] , since a > 0

= $\frac{1}{2}$ P [Z≤z]

=P [Z≤z]. F (a)

=P [Z≤z]. P [Y=0]

Similarly, it can be shown that,

P [Z≤z , Y=1]=P [Z≤z]. P [Y=1]

Hence,

P [Z≤z]. P [Y= y]= P [Z ≤z, Y=y]

Hence, Y and Z are independently distributed.

Q27. A bag contains a coin of value M and a number of other coins whose aggregate value is m. A person draws coins one at a time till the draws the coin of value M.

Find the value of his expectation.

ANS:- Let the coins be A, $B_1, B_2, \ldots, B_n$.

Value of A= M & value of $B_i = m_i$(say)

Such that $\sum_{i=1}^{n} m_i = m$.

Let Y_x be the value of the coins if x drawings are needed, x= 0, 1, 2, …., n+1.

And Y is the total value of the coins eventually.

Now, E(Y)=E {E(Y_x|X = x)}

Now, E (Y_x|X = 1)=M

E (Y_x|X = 2) =$\frac{M+m_1}{n}$ +$\frac{M+m_2}{n}$ + ….. + $\frac{M+m_n}{n}$

=M + $\frac{m}{n}$

E (Y_x|X = 3) = M + $\frac{2m}{n}$

In general, E (Y_x|X = x) = M + $\frac{(x-1)m}{n}$

Now, E (Y)= $\sum_{x=1}^{n+1}[M + (x-1)\frac{m}{n}]$.P (X=x)

Now, P (X= x)= P(X drawings are required to get the coin A)

=$\frac{n}{n+1} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n-1} \ldots \frac{n-x+1}{n-x+2} \cdot \frac{1}{n-x+1}$

=$\frac{1}{n+1}$

∴ E(Y)= $\sum_{x=1}^{n+1}[M + (x-1)\frac{m}{n}] \cdot \frac{1}{n+1}$

= $\frac{1}{(n+1)}[(n+1)M + \frac{m}{n} \cdot \frac{n(n+1)}{2}]$

= M + $\frac{m}{2}$.

So the required value of the expectation is (M + $\frac{m}{2}$).

Q28. Let X & Y be two joining distributed continuous random variable with joint PDF,

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-P^2}} \exp\left[-\frac{1}{2(1-P^2)}\{x^2 - 2P_{xy} + y^2\}\right],\ x\in\mathbb{R},\ y\in\mathbb{R}$$

I > find the marginal PDF of X

ii> Find the conditional PDF of Y for given X = x.

<u>Solution</u> :- i> $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$

$=\frac{1}{2\pi\sqrt{1-P^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2(1-P^2)}\{x^2 - 2P_{xy} + y^2\}\right]dy$

$=\frac{1}{2\pi\sqrt{1-P^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2(1-P^2)}\{(y-Px)^2 + (1-P^2)x^2\}\right]dy$

$=\frac{1}{2\pi\sqrt{1-P^2}}\, e^{-x^2/2} \int_{-\infty}^{\infty} \exp\left[-\frac{(y-Px)^2}{2(1-P^2)}\right] dy$

Since, $\frac{1}{\sqrt{2\pi}\sigma_Y} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma_Y{}^2}(y-\mu Y)^2\right] = 1$

Here, Y = Px and $\sigma^2{}_Y = (1-P^2)$

$\therefore \frac{1}{\sqrt{2\pi}\sqrt{(1-P^2)}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2(1-P^2)}(y-Px)^2\right] dy$

$\therefore \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2(1-P^2)}(y-Px)^2\right] dy = \sqrt{2\pi}(1-P^2)$

$\therefore f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, x∈ℝ,

$\therefore$ X~ N(0, 1).

ii> Coditional PDF of Y for Given X= x is

$$f_{\frac{y}{x}=x}\left(\frac{y}{x}\right) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\frac{1}{2\pi\sqrt{1-P^2}} e^{-x^2/2}\, e^{-\frac{(y-Px)^2}{2(1-P^2)}}}{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}}$$

$=\frac{1}{\sqrt{2\pi}(1-P^2)}\, e^{-\frac{(y-Px)^2}{2(1-P^2)}}$

i.e. $\frac{Y}{X}$=x~ N $(P_x, (\sqrt{1-P^2})^2)$, $\infty < y < \infty$

E $[\frac{Y}{X} = x] = \int_{-\infty}^{\infty} f_{XY}{}^{\left(\frac{y}{x}\right)dy}$ =Px.

Q29. Let x and Y have the circular normal distribution with zero mean, i.e. X & Y ~ N_2(0, 0, σ^2, σ^2, 0). Cosider a circle C and a square S of equal area both with ac (0, 0).

Prove that, P [(X , Y)∈s].

Solution:- The joint PDF of X & y is given by

F(x,y)=$\frac{1}{2\pi\sigma^2}\ e^{-\frac{1}{2\sigma^2}(x^2+y^2)}$, x∈ℝ, y ∈ ℝ, $\sigma > 0$

Let us consider a square S, with vertices (a, -a), (a, a), (-a, -a)

The area of the square =4a^2= s

Consider a circle C with radius = r, and the centre at (0, 0)

Area of C = πr^2

Hence, πr^2=4a^2[given]

⇒r =$\frac{2a}{\sqrt{\pi}}$

Therefore, a < r < $\sqrt{2a}$

Now, P [XY∈ S] =$\iint_{x,y\in S} f\ (x,y)dxdy$

= 4 $\int_0^a \int_0^a f\ (x,y)dxdy$ [By symetry]

P [X, Y∈ C]= $\iint_{x,y\in C} f\ (x,y)dxdy$

Now in the first quadrant,

P [X, Y∈ C] - P [X, Y∈ S]

= $\iint_{x,y\in A} f\ (x,y)dxdy$ - $\iint_{x,y\in B} f\ (x,y)dxdy$ [From the figure canceling the common region]

A = shaded region,

B = dotted region.

Now, if (x, Y)∈ A, then,

$x^2 + y^2$ <r^2

⇒- $\frac{(x^2+y^2)}{2\ \sigma^2}$ > - $\frac{r^2}{2\ \sigma^2}$

$\Rightarrow$ f(x, y)> $\frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$ ………………<i>

If (X, Y)∈ B

$x^2 + y^2 > r^2$

$\Rightarrow$ f(x, y)< $\frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$ ………………<ii>

From <i> & <ii> we get,

$\iint_{x,y\in A} f(x,y)dxdy > \iint_{x,y\in B} f(x,y)dxdy$

∴P [X, Y∈ C] > P [X, Y∈ S]

This inequality similarly holds for the other quadrants.

Q30. Show that, $\frac{1}{\sqrt{2\pi}} \int_0^a e^{-\frac{x^2}{2}} dx < \sqrt[\frac{1}{2}]{1 - e^{-\frac{2a^2}{\pi}}}$

Solution:- P (x, Y∈ S)=4 $\int_0^a \int_0^a \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}(x^2+y^2)} dxdy$

$= \frac{4}{2\pi} [\int_0^a e^{-\frac{x^2}{2}} dx]^2$

P (x, Y∈ C) = 4 $\int_0^{\pi/2} \int_0^r \frac{1}{2\pi} e^{-\frac{1}{2\sigma^2} R^2}$. RdRd0

$=\frac{4}{4}(1-e^{-\frac{r^2}{2}})$

∴ P (x, Y∈ C) > P (x, Y∈ S)

$\Rightarrow \frac{1}{4}\left(1 - e^{-\frac{r^2}{2}}\right) > \frac{1}{2\pi} [\int_0^a e^{-\frac{x^2}{2}} dx]^2$

$\Rightarrow$, $\frac{1}{\sqrt{2\pi}} \int_0^a e^{-\frac{x^2}{2}} dx < \sqrt[\frac{1}{2}]{1 - e^{-\frac{2a^2}{\pi}}}$

Q31. Let X and y be two r. v.'s with means zero, varience unity and correlation coefficient P, then S. T. E[Max(X^2, Y^2)]≤ 1 + $\sqrt{1 - P^2}$

Solution:- $\text{Max}(X^2, Y^2) + \text{Min}(X^2, Y^2) = X^2 + Y^2$

$\text{Max}(X^2, Y^2) - \text{Min}(X^2, Y^2) = X^2 - Y^2$

$$\text{Max}(X^2, Y^2) = \frac{1}{2}\ [\ (X^2 + Y^2) + \ |X^2 - Y^2|]$$

E [Max(X^2, Y^2)] = $\frac{1}{2}\ [E\ (X^2) + E(Y^2) + \ E|(X+Y)(x-Y)|]$

$=\frac{1}{2}\ [1 + 1\ E|(X+Y)(x-Y)|]$

By C – S inequality,

$$E^2[|(X+Y)(x-Y)|] \leq E(X+Y)^2 E(X-Y)^2$$

i.e. $E^2|X^2 - Y^2|\ \leq \big(2 + 2E(XY)\big)\big(2 - 2E(XY)\big)$

$\Rightarrow$ E $|X^2 - Y^2|\ \leq\ \sqrt[2]{1 - \ P^2}$

$\therefore$ E [max(X^2, Y^2)]$\leq 1 + \frac{1}{2}\ .\ \sqrt[2]{1 - \ P^2}$

$\leq 1 + \sqrt{1 - \ P^2}$

Q32. (a) S.T. for a r.s. $\boldsymbol{X_1, X_2, \ldots, X_n\ from\ n(\mu, \sigma^2)}$, show that,

$$\boldsymbol{\frac{\sqrt{\frac{n}{n-1}}\,(X_1 - \bar{x})}{\sqrt{\frac{(n-1)S^2 - \frac{n}{n-1}(X_1 - \bar{x}^2}{n-2}}} \sim t_{n-2}}$$

<u>Solution</u>:- (a) Let $Y_i = \sqrt{\frac{i}{i-1}}\ (\bar{x}_i - X_i) = \sqrt{\frac{i}{i-1}}(\frac{X_1 + X_2, \ldots, X_i}{i} - X_i)$

$=\sqrt{\frac{i}{i-1}}(\frac{X_1 + \ldots + x_{i-1} - (i-1)Xi}{i})$

$=\frac{X_1 + \ldots + x_{i-1} - (i-1)Xi}{\sqrt{i(i-1)}}$ $\forall$ i=2(1)n.

Let $Y_1 = \frac{1}{\sqrt{n}}\ X_1 + \cdots + \frac{1}{\sqrt{n}} X_n$.

Hence the transformation reduces Y=AX, where

A= $\frac{1}{\sqrt{n}}$ $\frac{1}{\sqrt{n}}$ $\frac{1}{\sqrt{n}}$ ………………….$\frac{1}{\sqrt{n}}$

$\frac{1}{\sqrt{1.2}}$ $\frac{1}{\sqrt{1.2}}$ 0 ……………0 is orthogonal

⋮ ⋮ ⋮ ………… ⋮

$\frac{1}{\sqrt{n(n-1)}}$ $\frac{1}{\sqrt{n(n-1)}}$ $\frac{1}{\sqrt{n(n-1)}}$ …..$\frac{(n-1)}{\sqrt{n(n-1)}}$ [Helmert's transformation]

The PDF of $(X_1, \ldots\ldots, X_n)$ is

F$(x_1, \ldots\ldots, x_n)$ = $(\frac{1}{\sigma\sqrt{2\pi}})^n e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{(X_i-\mu)^2}{\sigma^2}}$, $x_i \in$ R

Here $y^1 y = x^1 x$ and $y_1 = \sqrt{n}\bar{x}$

∴ |J|=1 is the jacobian of the transformation.

The PDF of $(y_1, \ldots\ldots, y)$ is

g $(y_1, y_{2,} \ldots\ldots, y_n)$= $(\frac{1}{\sigma\sqrt{2\pi}})^n e^{-\frac{+\sum_{i=1}^{n} y_i^2 - 2\mu\sqrt{n}\, y_1 n\mu^2}{2\sigma^2}}$

= $\{\frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{(y_1-\mu\sqrt{n})^2}{2\sigma^2}}\}\prod_{i=2}^{n}\{\frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{y_i^2}{2\sigma^2}}\}$

Hence $y_1 \sim$ N $(\mu\sqrt{n}\,, \sigma^2\ )$and $y_i \sim$ N $(0\,, \sigma^2)$ I = 1(1)n independently distributed.

Now, $\sum_{i=2}^{n}\frac{i}{i-1}\ (X_i - \bar{x}_i)^2 / \sigma^2$

$=\sum_{i=2}^{n}(\frac{y_i}{\sigma})^2$, which is the sum of squares of (n-1)

i.i. d. N (0, 1) R.V.'s , follows ${\lambda_{n-1}}^2$

$[y_i \sim$ N $(0\,, \sigma^2)\,, i = 2(1)n$

$\Rightarrow \frac{y_i}{\sigma} \sim$ N (0 ,1)]

$\Rightarrow\ \sum_{i=2}^{n}\frac{i}{i-1}(X_i - \bar{x}_i)^2\ \sim \sigma^2{\lambda_{n-1}}^2$

Note :- $y_1 = \sqrt{n}\,\bar{x} \sim$ N $(\mu\sqrt{n}\,, \sigma^2)$

(c) Consider the transformation

Y= A(X $-\mu$1), where

A= $\frac{1}{\sqrt{n}}$ $\frac{1}{\sqrt{n}}$ $\frac{1}{\sqrt{n}}$………………..$\frac{1}{\sqrt{n}}$

$\frac{-(n-1)}{\sqrt{n(n-1)}}$ $\frac{1}{\sqrt{n(n-1)}}$ $\frac{1}{\sqrt{n(n-1)}}$ … … … … … … … … .. $\frac{1}{\sqrt{n(n-1)}}$

a_{31} a_{32} a_{33} …………………….. a_{3n}

a_{n1} a_{n2} a_{n3} …………………….. a_{nn}

Note that y' y= (X_1- μ)' (X –μ)

$\Rightarrow \sum_{i=1}^{n} y^2{}_i = \sum_{i=1}^{n}(X_i - \mu)^2$

And |J=1|

The PDF of X is

$f_X(x) = (\frac{1}{\sigma\sqrt{2\pi}})^n . e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2}$, $x_i \in$ R

The PDF of Y is

$f_Y(y) = (\frac{1}{\sigma\sqrt{2\pi}})^n . e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y^2{}_i}$, $y_i \in$ R

$\Rightarrow y_i \sim$ N (0 , σ^2), i= 1(1) n

Here, $y_1 = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu) = \sqrt{n}$ (x̄- μ)

and $y_2 = \frac{-(n-1)(X_1 - \mu) + (X_2 - \mu) + \ldots + (X_n - \mu)}{\sqrt{n(n-1)}}$

$= \frac{n\,\bar{x} - nX_1}{\sqrt{n(n-1)}} = -\sqrt{\frac{n}{n-1}}\,(X_1 - \bar{x})$

Hence, $\sqrt{\frac{n}{n-1}}(X_1 - \bar{x}) = -y_2 \sim$ N (0 , σ^2)

And $\sum_{i=3}^{n} y^2{}_i = \sum_{i=1}^{n}(X_i - \mu)^2 - y^2{}_1 - y^2{}_2$

$=\{\sum_{i=1}^{n}(X_i - \mu)^2 - n(\bar{x} - \mu)^2\} - \frac{n}{n-1}(X_1 - \bar{x})^2$

$= \sum_{i=1}^{n}(X_i - \bar{x})^2 - \frac{n}{n-1}(X_1 - \bar{x})^2$

Therefore, $\sqrt{\frac{n}{n-1}}(\frac{X_1 - \bar{x}}{\sigma}) = -\frac{Y_2}{\sigma} \sim$ N(0, 1)

And, $\frac{\sum_{i=1}^{n}(X_i - \bar{x})^2 - \frac{n}{n-1}(X_1 - \bar{x})^2}{\sigma^2} = \sum_{i=3}^{n}(\frac{Y_i}{\sigma})^2$,

The sum of squares of (n-2) iid N(0, 1) R.V.'S, follows ${\lambda_{n-2}}^2$, independent 1 by defn of t- distn.

$$\frac{\sqrt{\frac{n}{n-1}}\left(\frac{X_1-\bar{x}}{\sigma}\right)}{\sqrt{\{\frac{\sum_{i=1}^{n}(X_i-\bar{x})^2-\frac{n}{n-1}(X_1-\bar{x})^2}{\sigma^2}\}\Big/_{(n-2)}}} \sim t_{n-2}$$

$$\Rightarrow \frac{\sqrt{\frac{n}{n-1}}\left(\frac{X_1-\bar{x}}{\sigma}\right)}{\frac{\{(n-1)S^2-\frac{n}{n-1}(X_1-\bar{x})^2\}}{(n-2)}} \sim t_{n-2}$$

Q33. Suppose (X, Y)~ BN(0, 0, 1, 1, P). S.T.

i) $\frac{X^2-2PXY+Y^2}{1-P^2} \sim \lambda_2{}^2$

ii) $M_z(t)= [\{1-(1+P)t\}\{1+(1-P)t\}]^{-1/2}$

<u>Solution :-</u>i) (X, Y)~ BN(0, 0, 1, 1, P)

∴ the joint PDF of (X, Y) is given by,

$f_{XY}(x,y)=\frac{1}{\sigma_x\sigma_y\sqrt{1-P^2}.2\pi}\ e^{-\frac{1}{2(1-P^2)}(X^2-2PXY+Y^2)}$; (x, y)∈ $\mathbb{R}^2$

Let, U= X+ Y

V= X- Y

∴ $|J|=\frac{1}{2}$, X= $\frac{U+V}{2}$, Y = $\frac{U-V}{2}$

Now, note that,

$$\frac{1}{1-P^2}\ \{\frac{(U+V)^2}{4}+\frac{(U-V)^2}{4}-\ P\,\frac{(U+V)(U-V)}{2}\}$$

$=\frac{1}{1-P^2}\ \{\frac{U^2+V^2}{4}-\ P\,\frac{(U^2-V^2)}{2}\}$

$=\frac{1}{2(1-P^2)}\{U^2-PU^2+V^2-PV^2\}$

$=\frac{1}{2(1+P)(1-P)}\{U^2(1-P)+V^2(1+P)\}$

$=\frac{U^2}{2(1+P)}+\frac{V^2}{2(1-P)}$, ______________<i>

∴ Joint PDF of U and V is given by,

$f_{UV}(u,v)$=$\frac{1}{\sqrt{2}\,\sqrt{2\pi}\,\sqrt{1+P}}\ e^{-\frac{1}{4(1+P)}u^2}.\frac{1}{\sqrt{2}\,\sqrt{2\pi}\,\sqrt{1+P}}\ e^{-\frac{1}{4(1+P)}v^2}$, (u,v)∈$\mathbb{R}^2$

∴ U and V are independent.

U ~ N(0, 2(1+P))

∴ $\frac{U}{\sqrt{2(1+P)}} \sim N(0,1) \Rightarrow \frac{U^2}{2(1+P)} \sim \lambda_1{}^2$

Similarly, $\frac{V^2}{2(1-P)} \sim \lambda_1{}^2$.

∴ $\frac{U^2}{2(1+P)} + \frac{V^2}{2(1-P)} \sim \lambda_1{}^2$ [By the reproductive property of λ^2-distribution]

ii) MGF of X, Y is given By,

$M_{XY}(t)$=E(e^{tXY})

= E[E(e^{tXY} | X)]

Y| X ~ N(P_X, (1-P^2))

∴ E $[e^{tX.PX} + \frac{1}{2}t^2X^2(1-P^2)]$

= E $[e^{tX.PX^2} + \frac{1}{2}P^2(1-P^2)X^2]$

= E $[e^{\{tP+\frac{1}{2}t^2(1-P^2)\}X^2}]$

= $\frac{1}{[1-2(tP+\frac{1}{2}t^2(1-P^2))]^{1/2}}$ [∵X^2~$\lambda_1{}^2$]

= $\frac{1}{[1-2tP-t^2(1-P^2)]^{1/2}}$

= $\frac{1}{\sqrt{(1-tP)^2-t^2}}$

= $\frac{1}{\sqrt{\{(1-P)t+1\}\{(1-t(1+P)\}}}$

= $[\{1-(1+\text{P})\text{t}\}\,\{1+(1-\text{P})\text{t}\}]^{-1/2}$

Q34. X ~ R(0, 1) find

i) the distn of $X_{(r)}$.

ii) The m.g.f. , mean (E($X_{(r)}$)) and variance (var($X_{(r)}$))

ANS:- i) If X ~ R (0, 1), then the p.d.f of X is given by,

F(x)=1,   0 < x < 1

The distn function of X is given by,

F(x)= $\int_0^x dx$=x

∴ The PDF of r^{th} order statistic is given by,

g(x) = $\frac{n!}{(r-1)!(n-r)!} x^{r-1} (1-x)^{n-r}$, 0< x < 1

∴ $X_{(r)}$ ~ B (r, n-r+1)

ii) Let $X_{(r)}$ be denoted as U, then

U ~ B(r-1, n-r)

$M_U(t)$= E(e^{Ut})

=E [1+Ut+$\frac{U^2t^2}{2!}+\cdots$]

= $\sum_{r=0}^{\infty} \frac{t^r}{r!} E(U^r)$

=$\sum_{r=0}^{\infty} \frac{t^r}{r!} .{\mu_r}^1$ [∵${\mu_r}^1$ = rth order raw moment about zero]

E(U)=$\frac{1}{B(r,n-r+1)} \int_0^1 u.u^{r-1}(1-u)^{n-r}\, du$, 0<x <1

=$\frac{B(r+1,n-r+1)}{B(r,n-r+1)}$

=$\frac{r}{n+1}$.

E (U^2)=$\frac{r(r+1)}{(n+1)(n+2)}$

V(U) =$\frac{r(r+1)}{(n+1)(n+2)}$ - $\frac{r^2}{(n+1)^2}$ = $\frac{r}{n+1}$ [$\frac{r+1}{n+2} - \frac{r}{n+1}$]

=$\frac{r(n-r+1)}{(n+1)^2\,(n+2)}$

Q35. If X_1, X_2 be a random sample of size 2 drawn from a population having p.d.f. f(x)=λe $^{-\lambda x}$, x> 0, λ>0. Then find the distn of the sample range. Is the distn independent from the sampling distribution of simple AM?

ANS:- X_1, X_2 be a random sample drawn from a population with pdf f(x)= λe $^{-\lambda x}$, x>0, λ> 0.

Let us consider the following transformation

$(X_1, X_2) \rightarrow (X_{(1)}, X_{(2)})$, where $X_{(i)}$=ith order statistic.

∴ Joint distn of $X_{(1)}, X_{(2)}$ is given by,

$f_{X_{(1)}, X_{(2)}}{}^{(x_1, x_2)} = 2\lambda^2 \, e^{-\lambda(X_{(1)} + X_{(2)})}$

Let us define a variable,

$U_i = X_i - X_{(i-1)}$ ∀ i=1, 2.

$U_1 = X_{(1)}$ [Assuming$X_{(0)} = 0$]

$U_2 = X_{(2)} - X_{(1)}$

∴ $X_2 = u_1 + u_2$

∴ $|J| = |J(\frac{X_{(1)}, X_{(2)}}{u_1, u_2})| = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$

∴ Joint pdf of u_1, u_2 is given by,

$f_{U_1, U_2}{}^{(u_1, u_2)} = 2\lambda^2 e^{-\lambda(2u_1 + u_2)}$

=2 λ .$e^{-2\lambda u_1}$, λ .$e^{-2\lambda u_2}$, (u_1, u_2)>0

∴ U_1 , U_2 are independently distributed,

∴ sample range(R)= $X_{(2)} - X_{(1)}$

$= u_2$

∴ E(R)=E(u_2)= $\lambda \int_0^{\infty} u \, e^{-\lambda u_2} \, du$

$= \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$

∴ PDF of sample range (R) is f_R(R)=λ $e^{-\lambda R}$, R> 0

Now, simple AM = $\frac{X_{(1)} + X_{(2)}}{2} = \frac{2u_1 + u_2}{2} = u_1 + \frac{1}{2} u_2$= Z, say,

∴ Joint PDF of (Z, u_2) is given by,

f_{Z, u_2}(Z, u_2)= $2\lambda^2 \, e^{-2\lambda Z}$, Z>0

So, the distn of sample range & simple AM are different.

Q36. F(x, y) be a joint distribution function of X and Y. G(ξ,)be a function ∋ξ= max(X, Y). Show that G(x, y){$\boldsymbol{F(x,x) if\ x} < y$

F(x, y)+F(x,x)-F(y, y) if x≥ y

Solution:- G(x, y)= P [Max(X, Y)≤x, Min(X, Y)≤ y] if x < y

= P[P [Max(X, Y)≤x, Min(X, Y)≤ x]

=P [X≤ x , Y ≤ x]

= F(x, x)

Now, G(x, y)= P [Max(X, Y)≤x, Min(X, Y)≤ y] if x ≤ y

= P[$X_{(2)}$ ≤x, $X_{(1)} \leq y$]; A= $X_{(2)}$ ≤x, B=$X_{(1)} \leq y$

=P (A) –P(A ∩BC)

= P[$X_{(2)}$ ≤x] –[P[$X_{(2)}$ ≤x, $X_{(1)} \geq y$]

= P [X≤ x, Y≤ x]- P [y ≤ $X_{(1)} \leq X_{(2)} \leq x$]

= F(x,x)-(F(y,y)-F(x, y))

= F(x,x) +(F(x ,y)-F(y , y)).

Q37. f(x,y)= {$\boldsymbol{\frac{1}{\pi}\ if\ x^2 + y^2 \leq 1}$ $(\boldsymbol{a})$Are X and Y uncorrelated?

=0 if $\boldsymbol{x^2 + y^2} > 1$ (b)Are X and y independent?

Solution:- f(x,y)= {$\frac{1}{\pi} I\ x^2 + y^2 \leq 1$

$f_X(x) = \frac{1}{\pi}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dX = \frac{2}{\pi}\sqrt{1-x^2}$; -1 < x < 1

$f_Y(y)$=$\frac{2}{\pi}\sqrt{1-y^2}$; -1 < y < 1

E(X)= $\frac{2}{\pi}\int_{-1}^{1} x\sqrt{1-x^2}$ dx = 0 [∵ the function is odd]

Similarly, E(Y)=0

E(XY)=$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} xy\ \frac{1}{\pi}\ dxdy = 0$ [∵ $\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{x}{\pi} dx = 0$]

∴ X and Y are uncorrelated.

Note that, f(x,y)=$\frac{4}{\pi}\sqrt{(1-x^2)(1-y^2)}$ ≠ f(x). f(y).

∴ X and Y are not independent.

Q38. X_1, ………, X_n , …… be i.i.d r. v. 's satisfying P [X_1=2^j]= $\frac{1}{2^j}$,

j = 1, 2, 3, ….. show that WLLN dose not hold for {X_n}.

Solution:- WLLN holds iff $\underset{n\to\infty}{Lt}[|x| > n] = 0$

Let, $2^k \leq n \leq 2^{k+1}$

i.e. n=2^k +r

P[X > n]=P [$X \geq 2^{k+1}$]= P [$X = 2^{k+1}$]+ $X \geq 2^{k+2}$+ ……

=$\frac{1}{2^{k+1}}$+$\frac{1}{2^{k+2}}$ + …….

= $\frac{1}{2^{k+1}}$.(1+ $\frac{1}{2}$ + $\frac{1}{4}$+ …..)= $\frac{1}{2^k}$

∴ nP [X > n]= $\frac{2^{k+r}}{2^k}$

= 1+$\frac{r}{2^k}$

∴ $\underset{n\to\infty}{Lt}\, nP[x > n]$= $\underset{k\to\infty}{Lt}\,(1+\frac{r}{2^k})$=1.

∴ WLLN does not hold

Q39. There are 10 balls in an urn numbered 1 through 10.You randomly select 3 of those balls. Let the random variable Y denotes the maximum of the three numbers on the extracted balls. Find the probability mass function of y. You should simplify your answer to a fraction that does not involve binomial coefficients. Then calculate:

P [Y ≥7].

Solution:- The random variable Y can take the values in the set {3, 4, …. , 10}. For any I, the triplet resulting in Y attaining the value i must consist of the ball numbered i and a pair of balls with lower numbers. So ,

Pi =P[Y =i]= $\frac{\binom{i-1}{2}}{\binom{10}{3}}$= $\frac{\frac{(i-1)(i-2)}{2}}{\frac{10.9.8}{3.2.1}}$ = $\frac{(i-1)(i-2)}{240}$

Since the balls are numbered 1 through 10, we have

P [Y ≥7] =P [Y= 7] + P [Y= 8] + P[Y=9] + P [Y= 10]

So, P [Y ≥7]= $\frac{6.5}{240}+\frac{7.6}{240}+\frac{8.7}{240}+\frac{9.8}{240}$

$=\frac{5}{6}$.

Q40. The number of misprints per page of text is commonly modeled by a poisson distribution . It is given that the parameter of this distribution is λ = 0.6 for a particular book. Find the probability that there are exactly two misprints on a given page of the book. How about the prob. that there are two or more misprints?

Solution:- Let X denote the random variable which stands for the number of misprints on a given page. Then

P [X =2]= $\frac{0.6^2}{2!}\,e^{-0.6}\approx 0.0988$

P [X≥ 2] = 1- P [X < 2]

= 1- P [X=1] – P[X=0]

=1-$e^{-0.6}$ - 0.6$e^{-0.6}$

≈ 0.122.

Q41. The unit interval (0,1)is divided into two subintervals picking a point at random from inside the interval. Denoting by Y and Z, the lengths of yhe larger and the shorter subintervals respectively. Show that Y/Z does not have finite expectations.

Solution:- Let X ∼ U(0, 1) and U be the length of the shoter of the intervals (0, X) and (X, 1); i.e. Z = min(X, 1-X)

And let, Y = 1- U.

We can write Y and Z as Y = Y(X), Z = Z(X); as functions of X ∼ U (0, 1).

To show the expectation is infinite; we need to show that

$\int_1^0 \frac{Y(x)}{Z(x)}\,dx=\infty$

Now, Z = min (X, 1-X) , Y = max(X, 1-X)

But, $\int_1^{\frac{1}{2}} \frac{Y(x)}{Z(x)}\,dx = \int_1^{\frac{1}{2}} \frac{\max\,(X,1-X)}{\min\ (X,1-X)}\,dx = \int_1^{\frac{1}{2}} \frac{1-x}{x}\,dx = \infty$

One get the same thing on the interval [$\frac{1}{2}$, 1] as well after a substitution, so the integral on [0, 1] is ∞.

Aliter:- let T (x)= $\frac{\max(X,1-X)}{\min(X,1-X)}$

To calculate P [T (X)≤], for some t, If t < 1, then trivially we get 0.

Ow, P [T (X)≤]= P [$\frac{1-X}{X} \leq t$, X∈ (0, $\frac{1}{2}$)] + P [$\frac{X}{1-X} \leq t$, X∈ ($\frac{1}{2}$, 1)]

= 2P [X≥ $\frac{1}{1+t}$, X∈ (0, $\frac{1}{2}$)]

=2 $\int_{1/_{1+t}}^{1/2} dx$ = 1- $\frac{2}{t+1}$.

Differentiating, we get f(t) = $\frac{2}{(t+1)^2}$ for t≥ 1.

Now, $\int_1^{\infty} \frac{2}{(t+1)^2}\, dt = \infty$.

Q42. Let $X_1, X_2 \sim$ R(0, 1). Show that____________

$\boldsymbol{U_1} = \sqrt{-21nX_1}\,\cos(2\pi X_2)$

$\boldsymbol{U_2} = \sqrt{-21nX_1}\,\sin(2\pi X_2)$

Are standard normal variables.

Solun:- The PDF of (X_1, X_2) is $f_{X_1,X_2}(x_1, x_2) = f(x) = \begin{cases} 1 & , \ 0 < x_1, x_2 < 1 \\ 0 & , \ ow \end{cases}$

Here, $u_1 = \sqrt{-21nx_1}\,\cos(2\pi x_2)$

$u_2 = \sqrt{-21nx_1}\,\sin(2\pi x_2)$

∴ ${u_1}^2 + {u_2}^2$ = -21nx_1

⇒$x_1 = e^{-\frac{1}{2}({u_1}^2 + {u_2}^2)}$

And tan($2\pi x_2$)= $\frac{u_2}{u_1}$

⇒ $x_2 = \frac{1}{2\pi}\tan^{-1}\frac{u_2}{u_1}$

Note that, 0 < x_1 < 1

$\Rightarrow$ -21nx_1 > 0 , 0<$2\pi x_2$ <2π

$\Rightarrow \sqrt{-21nx_1}$ > 0 , -1 ≤ $\cos(2\pi x_2)$, $\sin(2\pi x_2)$ ≤ 1

$\Rightarrow u_1$, $u_2 \in \mathbb{R}$

The Jocobian is J = $\begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_1} \end{vmatrix}$

$$=\begin{vmatrix} e^{-\frac{1}{2}(u_1{}^2+u_2{}^2)}.(-u_1) & e^{-\frac{1}{2}(u_1{}^2+u_2{}^2)}.(-u_2) \\ \frac{1}{2\pi\{1+(\frac{u_2}{u_1})^2\}}.(-\frac{u_2}{u_1}) & \frac{1}{2\pi\{1+(\frac{u_2}{u_1})^2\}}.\frac{1}{u_1} \end{vmatrix}$$

$$= \frac{e^{-\frac{1}{2}(u_1{}^2+u_2{}^2)}}{2\pi\{1+(\frac{u_2}{u_1})^2\}} \begin{vmatrix} -u_1 & -u_2 \\ -\frac{u_2}{u_1{}^2} & \frac{1}{u_1} \end{vmatrix}$$

$$= -\frac{1}{2\pi} . e^{-\frac{1}{2}(u_1{}^2+u_2{}^2)}$$

The PDF of (u_1 , u_2) is

$$f_{U_1,U_2}(u_1, u_2) = 1 . \left| -\frac{1}{2\pi} . e^{-\frac{1}{2}(u_1{}^2+u_2{}^2)} \right| . (u_1 , u_2) \in R^2$$

$$= \frac{1}{\sqrt{2\pi}}.e^{-\frac{1}{2}u_1{}^2} . \frac{1}{\sqrt{2\pi}}.e^{-\frac{1}{2}u_2{}^2} , (u_1 , u_2) \in \mathbb{R}^2$$

$$= f_{u_1}(u_1).f_{u_2}(u_2) , u_1 , u_2 \in \mathbb{R}$$

Hence, $U_1, U_2 \sim$ N(0, 1).

Q43. Let X, Y ~ N(0, 1). Show that U = $\frac{X}{Y}$ has a standard Cauchy distribution. What would be the distn of $\frac{X}{|Y|}$?

Soln:- Here, $f_{X,Y}(x, y) = \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}$, (x, y) $\in \mathbb{R}^2$

Let, U=$\frac{X}{Y}$ and V = y.

$\therefore$ u = $\frac{x}{y}$, v= y [$-\infty < u < \infty$, $-\infty < v < \infty$]

$\Rightarrow$ x= uv , y=v

$\therefore$ J = $\begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix}$= v = $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

Clearly, (u,v) $\in \mathbb{R}^2$

The PDF (U, V) is ___

$f_{U,V}(u, vy)= \frac{1}{2\pi} e^{-(1+u^2)\frac{v^2}{2}} |v|$, (u,v) $\in \mathbb{R}^2$

The PDF of U is

$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u, v)\, dv = \int_{-\infty}^{\infty} \frac{1}{2\pi}\ e^{-(1+u^2)\frac{v^2}{2}} |v| dv$

Or, =[2 $\int_0^{\infty} \frac{v}{2\pi} e^{-\frac{1}{2}v^2(1+u^2)} dv$] $\quad = \frac{2}{\pi} \int_0^{\infty} e^{-(1+u^2)\frac{v^2}{2}} v dv$

[$\int_0^{\infty} \frac{1}{\pi}\ \frac{1}{(1+u^2)}\ . e^{-Z} dZ \frac{1}{2} v^{2(1+u^2)=Z]}$ $\quad = \frac{1}{\pi} \int_0^{\infty} e^{-(1+u^2)Z}\, dZ$ [where, Z=$\frac{v^2}{2}$, ⇒dZ = vdv]

[$\frac{1}{\pi(1+u^2)}[-e^{-Z}]$] $\quad = \frac{1}{\pi} . \frac{\Gamma(1)}{(1+u^2)}$, u ∈ $\mathbb{R}$

[$\frac{1}{\pi(1+u^2)}$] $\quad = \frac{1}{\pi(1+u^2)}$, u ∈ $\mathbb{R}$

Hence, U=$\frac{X}{Y}$ ~ C(0, 1)Distn

Let, w = $\frac{x}{|Y|}$, The PDF of w is $f_W(w)$= P [W≤ w/Y < 0]P [Y < 0]+ P [W≤ w/Y > 0]P [Y > 0]

= $\frac{1}{2}$ {P [$\frac{X}{-Y} \leq w$] + P [$\frac{X}{Y} \leq w$]}

= $\frac{1}{2}$ {P[-U≤ w] + P [U ≤ w]}

= $\frac{1}{2}$ .2 .P [U≤ w] [∵ U~ C (0, 1)is symmetrical about '0']

[⇒$f_U(-u) = f_U(u)$

⇒ u and –U have identical distribution]

$\therefore f_W(w) = f_U(w)$ ∀ w

⇒ W = $\frac{X}{|Y|}$~ C (0, 1).

Q44. If X, Y ~ N(0, 1). Find the distns of U = $\sqrt{X^2+Y^2}$ and V = $\frac{X}{Y}$

Solution:- $f_{x,Y}(x,y) = \frac{1}{2\pi} . e^{-\frac{1}{2}(x^2+y^2)}$, (x, y)∈ $\mathbb{R}^2$

Note that, u= $\sqrt{x^2+y^2}$, v = $\frac{x}{y}$

⇒ u = |y|. $\sqrt{1+v^2}$, x= vy

⇒ x = ± $\frac{uv}{\sqrt{1+v^2}}$, y = ± $\frac{u}{\sqrt{1+v^2}}$

Let, $x_1 = \frac{uv}{\sqrt{1+v^2}}$, $y_1 = \frac{u}{\sqrt{1+v^2}}$

Then for a pair (U, V), there are two points of (x, y):

$(x_1, y_1), (-x_1, -y_1)$

The transformation is not one – to one,

Clearly, 0 < u < ∞ , v ∈ ℝ

$$\text{Now, } J_1 = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{v}{\sqrt{v^2+1}} & \frac{u}{(\sqrt{v^2+1})^{3/2}} \\ \frac{1}{\sqrt{v^2+1}} & \frac{uv}{(\sqrt{v^2+1})^{3/2}} \end{vmatrix}$$

$= -\frac{u}{1+v^2} = J_2$

Hence, The PDF of (U, V)is

$$f_{U,V}(u,v) = \begin{cases} f_{x,Y}(x_1,y_1)|J_1| + f_{x,Y}(-x_1,-y_1)|J_2|, & if\ 0 < u < \infty, -\infty < v < \infty \\ 0 & , ow \end{cases}$$

$$= \begin{cases} \frac{2}{2\pi}\ e^{-\frac{u^2}{2}} . \left|-\frac{u}{1+v^2}\right|, & if\ 0<u<\infty, and\ -\infty<v<\infty \\ 0 & , ow \end{cases}$$

$$= \begin{cases} \left(ue^{-\frac{u^2}{2}}\right) . \frac{1}{\pi(1+v^2)}, & if\ 0 < u < \infty\ and\ v \in \mathbb{R} \\ 0 & , ow \end{cases}$$

Hence, U = $\sqrt{X^2+Y^2}$ has the PDF

$$f_U(u) = \begin{cases} u\, e^{-\frac{u^2}{2}}, & 0 < u < \infty \\ 0 & , ow \end{cases}$$

and V ~ Cauchy (0, 1), independently.

Q45. If X, Y ~ N(0, 1), Find the distn of U= $\boldsymbol{\frac{XY}{\sqrt{X^2+Y^2}}}$**, and V =** $\boldsymbol{\frac{X^2-Y^2}{\sqrt{X^2+Y^2}}}$**.**

Solution:- $f_{x,Y}(x,y)=\frac{1}{2\pi}\, e^{-\frac{X^2+Y^2}{2}}$, (x, y)∈ $\mathbb{R}^2$

Let, x= r$\cos\theta$, y = r$\sin\theta$,

Here, 0< r<∞, 0< θ< 2π,

∴ J = r,

The PDF of (r, θ)is ______

$$g(r,\theta)=\begin{cases} re^{-\frac{r^2}{2}}.\frac{1}{2\pi} ,0<r<\infty \ and \ 0<\theta<2\pi \\ 0 \quad , \qquad ow \end{cases}$$

Here, u= r$\sin\theta\cos\theta$ = $\frac{r}{2}sin^2\theta$

And v= r$\cos 2\theta$

Clearly, (U, V) ∈ $\mathbb{R}^2$

$$J_1=\frac{\partial(\mathrm{r},\theta)}{\partial(\mathrm{u},\mathrm{v})}=\frac{1}{\frac{\partial(\mathrm{u},\mathrm{v})}{\partial(\mathrm{r},\theta)}}=\frac{1}{\begin{vmatrix}\frac{1}{2}sin^2\theta & \mathrm{r}\cos 2\theta \\ \cos 2\theta & -2sin\theta\end{vmatrix}}=\frac{1}{r}=J_2$$

Here, $(2u)^2+v^2=r^2$ [a pair(u,v)is a obtained, for two pairs: (r, θ), (r, θ+2π). The transformation is not one-to-one]

⇒ r = $\sqrt{4u^2+v^2}$

The PDF of (U, V)is

$$f_{UV}(u,v)=\frac{2\,.e^{-\frac{4u^2+v^2}{2}}}{2\pi}.(\sqrt{4u^2+v^2})\left|\frac{1}{\sqrt{4u^2+v^2}}\right|, (u,v)\in\mathbb{R}^2$$

$$=\frac{1}{\frac{1}{2}\sqrt{2\pi}}\,.\,e^{-\frac{u^2}{2.\frac{1}{4}}}.\frac{1}{\sqrt{2\pi}}\,.\,e^{-\frac{v^2}{2}}\ ;\ (u,v)\in\mathbb{R}^2$$

= $f_U(u).f_V(u)$, u, v ∈ $\mathbb{R}$

Hence, U ~ N (0, $\frac{1}{4}$) and V ~ N (0, 1), independently.

Q46. Let $X_1, X_2 \sim$ r (0, 1). Find out CDF and hence the PDF of $X_1 + X_2$. How should the above result be modified in case $X_1, and\ X_2 \sim$ R (a, b)?

Solution:- $f_U(u)$ = P [U≤ u]

= P [$X_1 + X_2 \leq$ u]

=$\iint_{X_1 + X_2}(x_1, x_2)dx_1, dx_2$

Here, U = $X_1 + X_2$ takes values between 0 and 2.

Note that for 0< u <1,

P [U≤ u]= P [$X_1 + X_2 \leq u$]

= $\frac{Area\ of\ the\ region\ A}{Area\ of\ the\ samp\ le\ space\ (\Omega)}$

[Using the concept of geometric probability,

As (X_1, X_2) is uniformly distributed

Over Ω, here,

Ω = {(X_1, X_2) : 0< X_1, X_2< 1}

And A ={(X_1, X_2) : $X_1 + X_2 \leq u$}≤ Ω]

∴ P [U≤ u] = $\frac{\frac{1}{2}u^2}{1^2} = \frac{1}{2}u^2$, for 0< u< 1.

For, 1 ≤ u< 2 , ___

P [U ≤ u]= P [$X_1 + X_2 \leq u$]

= $\frac{Area\ of\ the\ region\ A}{Area\ of\ the\ sample\ space\ (\Omega)}$

= $\frac{1^2 - \frac{1}{2}(2-u)^2}{1^2}$

= 1 -$\frac{1}{2}(2-u)^2$

Hence the CDF of U is___ $F_U(u)= \begin{cases} 0, u \leq 0 \\ \frac{1}{2}u^2, 0 < u < 1 \\ 1 - \frac{1}{2}(2-u)^2, 1 \leq u < 2 \\ 1, u \geq 2 \end{cases}$

And the PDF of U is _____ $F_U(u)=\begin{cases} u, 0 < u < 1 \\ 2-u, 1 \leq u < 2 \\ 0, ow \end{cases}$

<u>Modification</u>:- $X_i \sim$ R (a, b) , i=1, 2

$\Rightarrow U_i = \frac{X_i - a}{b-a} \sim$ R (0, 1), i=1, 2.

Q47. Let $X_1, X_2 \sim$ R (0, 1). Find out CDF and PDF of

i> $|X_1 - X_2|$, ii>$X_1 X_2$

Solution:-

i> Let U=$|X_1 - X_2|$

Note that U takes values between 0 and 1.

For, 0< u< 1,

P [U ≤ u]= P [$|X_1 - X_2| \leq$ u]

= P [-u≤ $X_1 - X_2 \leq$ u]

=$\frac{Area\ of\ the\ shaded\ region}{Area\ of\ the\ sample\ space\ (\Omega)}$

[Using the concept of Geometric

Probability as (X_1, X_2) is

Uniformly distributed over Ω]

P [U ≤ u]= $\frac{1^2 - \frac{1}{2}(1-u)^2}{1^2}$

= 1- $(1-u)^2$

Hence,the CDF of u is

$f_U(u)=\begin{cases} 0, u \leq 0 \\ 1-(1-u)^2, 0 < u < 1 \\ 1, u \geq 1 \end{cases}$

And the PDF of U is

$f_U(u)=\begin{cases} 2(1-u), 0 < u < 1 \\ 0, ow \end{cases}$

ii> Let U= $X_1 X_2$

Then U takes value between 0 and 1.

For, 0< u< 1:

P [U ≤ u]= P [$X_1 X_2 \leq$ u]

$= \frac{Area\ of\ the\ shaded\ region}{Area\ of\ the\ sample\ space\ (\Omega)}$

$=\frac{uX_1+\int_u^1 x_2 dx_1}{1^2}$

$=u + \int_u^1 \frac{u}{x_1}\ dx_1$

= u+ u [1nx_1]1,u =u(1-|nu)

Q48. X and Y ~ R (0, 1) X & Y are independent,

i> X+Y ~ ?

ii> X- Y ~?
iii> XY ~?
iv> $\frac{X}{Y}$ ~?
v> $|X - Y|$~?

ANs:-i) Z= X+Y

0< X, Y<1

⇒ 0< Z < 2.

Distribution function of Z is

$F_Z(z)$= P [z≤ z]

= P[Y≤z-X]

$$= \begin{cases} 0, z \leq 0 \\ \frac{1}{2}z^2, if\ 0 < z < 1 \\ 1 - \frac{1}{2}(2-z)^2, if\ 1 < z < 2 \\ 1 \quad , \ if\ z \geq 2 \end{cases}$$

PDF of Z is, _____

$$F_Z(z)=\begin{cases} z\,, if\ 0<z<1 \\ 2-z\,, if\ 1<z<2 \\ 0\quad, \qquad ow \end{cases}$$

ii) Z = X- Y

0< X, Y< 1

⇒ -1< z< 1

$F_Z(z)$ = P [X-Y ≤z]

=P [Y ≥ $X-z$]

$$=\begin{cases} 0\,, if\ z\le\ -1 \\ \frac{1}{2}(z+1)^2, if -1<z<0 \\ 1-\frac{1}{2}(1-z)^2, if\ 0<z<1 \\ 1\quad,\ \ z\ge 1 \end{cases}$$

PDF of Z is, _________

$$F_Z(z)=\begin{cases} z+1\,, if -1<z<0 \\ 1-z\,, if\ 0<z<1 \\ 0\quad,\quad ow \end{cases}$$

iii) Z=XY

0< X, Y< 1 ⇒ 0< XY <1.

⇒ 0<x < 1

$F_Z(z)$= P [XY ≤ z]

$$=\begin{cases} 0\ \ , if\ z\ \le 0 \\ \int_0^1\int_0^z dxdy+\int_z^1\int_0^{z/x} dxdy\ \ , if\ 0<z<1 \\ 1\,,\quad if\ z\ge 1 \end{cases}$$

= z+$\int_z^1 \frac{z}{x}\,dx$, if 0< z< 1

= z + z [|n|-|nz]

= z- z|nz = z (1-|nz). , if 0< z<1

∴ CDF of z is, ______

$$F_Z(z)=\begin{cases} 0 \;\;, \; if\; z \le 0 \\ z\,(1-|nz), if\; 0<z<1 \\ 1 \;\;\;, if\; z \ge 1 \end{cases}$$

PDF of z is, ____

$$F_Z(z)=\begin{cases} -1nz\,, if\; 0<z<1 \\ 0 \qquad\quad , ow \end{cases}$$

iv) Z= $\frac{X}{Y}$

0< X, Y< 1

⇒ 0<$\frac{X}{Y}$ < ∞

∴ 0< Z <∞

P [Z ≤ z]

=P [$\frac{X}{Y}$ ≤ z]

= P [Y ≥ ½ X]

$$=\begin{cases} 0\,, if\; z \le 0 \\ \frac{1}{2}z\,, if\; 0<z<1 \\ 1-\frac{1}{2z}, if\; z \ge 1 \end{cases}$$

$$\therefore F_Z(z)=\begin{cases} \frac{1}{2}, if\; z \le 0 \\ \frac{1}{2z^2},\; if\; z\; \ge 1 \\ 0 \quad\;, \; ow \end{cases}$$

v) Z = $|X-Y|$

0< X, Y< 1

⇒ 0<$|X-Y|$ < 1

P [Z ≤z]

= P [X –Z ≤Y ≤ X+Z]

$$=\begin{cases} 0\,, if\ z \le 0 \\ 1-(1-z)^{2\,, if\ 0<z<1} \\ 1, if\ z\ \ge 1 \end{cases}$$

$$\therefore f_Z(z) = \begin{cases} 2(1-z)\,, if\ 0<z<1 \\ 0\ \ ,\ \ \ \ ow \end{cases}$$

Q49. X ~ R(0, a)

Y ~ R(0, b)

X and Y are independent & a> b

i> X+Y ~ ?

ii> X- Y ~?
iii> XY ~?
iv> $\frac{X}{Y}$ ~?
v> $|X-Y|$~?

ANS :- i) X + Y= Z,

0< x < a, 0<y < b

⇒ 0< x+y< a+b

∴ P [Z ≤z]= P [Y≤z-X]

$$= \begin{cases} 0\,, z \le o \\ \frac{1}{ab} \times \frac{1}{2} z^2\,, if\ 0<z\ \le b \\ \frac{1}{ab} \times \frac{b}{2} \times (2z-b), b<z\ \le 0 \\ \frac{ab-(\frac{a+b-z)^2}{2}}{ab}\ a<z<a+b \\ 1, if\ z\ \ge a+b \end{cases}$$

$$=\begin{cases} \frac{z}{ab}\ , a<z\ \le b \\ \frac{1}{a}, b<z \le a \\ \frac{a+b-z}{ab}\ , a<z<a+b \\ o\ \ ,\ \ ow \end{cases}$$

ii) X- Y = z

P [Z ≤ z]

= P [X-Y ≤ z]

= P [Y ≥ X- z]

$$= \begin{cases} 0 \,, z \le -b \\ \frac{1}{2ab}(b+z)^2 \,, -b < z \le 0 \\ \frac{1}{2ab}\,(b+2z)b, 0 < z \le a-b \\ 1 - \frac{1}{2ab}(a-z)^2 \,, a-b < z < a \\ 1 \quad , \quad z \ge a \end{cases}$$

$$\therefore f_z(z) = \begin{cases} \frac{1}{ab}(b+z), -b < z \le 0 \\ \frac{1}{a} \,, 0 < z \le a-b \\ \frac{a-z}{ab} \,, a-b < z < a \\ 0 \quad , ow \end{cases}$$

iii) Z= XY , 0< z < ab

P [Z ≤ z]

= P [Y ≤ $\frac{z}{X}$]

$$= \begin{cases} 0 \quad , z \le 0 \\ \frac{1}{ab}\left[z + \int_{\frac{z}{b}}^{a} \frac{Z}{X} dx\right], 0 < z < ab \\ 1 \quad , \quad z \ge ab \end{cases}$$

$$\therefore f_z(z) = \begin{cases} [\,1 - b + 1nba - 1nz]\,\frac{1}{ab}, 0 < z < ab \\ 0 \quad , \quad ow \end{cases}$$

iv) Z = $\frac{X}{Y}$, 0< z

P [$\frac{X}{Y} \le z$]

P[Y≥ $\frac{X}{z}$]

$$= \begin{cases} 0 \quad , \quad z \le 0 \\ \frac{1}{2ab} \times bz \times b \,, 0 < z < 1 \\ 1 - a.\frac{a}{z}.\frac{1}{2ab} \,, 1 \le z \end{cases}$$

$$\therefore f_z(z) = \begin{cases} \frac{b}{2a}, 0 < z < 1 \\ \frac{a}{2b}\left(\frac{1}{z^2}\right), z \geq 1 \\ 0, \quad ow \end{cases}$$

v) Z = |X- Y| , 0< z < a

P [X-z ≤Y ≤ X + z]

$$= \begin{cases} 0, z \leq 0 \\ 1 - \frac{1}{2}(b-z)^2 - \frac{1}{2}(a-z)^2, 0 < z < a \\ 1, \quad z \geq a \end{cases}$$

$$\therefore f_z(z) = \begin{cases} a + b - 2z, 0 < z < a \\ 0, \quad ow \end{cases}$$

Q50. let X_1, X_2, X_3 be iid RV's with PDF

F(x) = $\begin{cases} e^{-x}, x > 0 \\ 0, ow \end{cases}$

Show that ⟶ $Y_1 = X_1 + X_2 + X_3$

$Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}$

$Y_3 = \frac{X_1}{X_1 + X_2}$ **, are independently distributed.**

Identify their distribution.

Solution :- $f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \begin{cases} e^{-(x_1+x_2+x_3)}, if\ x_i > 0\ \forall\ i = 1,2,3 \\ 0, \quad ow \end{cases}$

Here, $x_1 + x_2 + x_3 = y_1$

$x_1 + x_2 = y_1 y_2$ $\quad [\because x_1 + x_2 < x_1 + x_2 + x_3 \Rightarrow \frac{x_1+x_2}{y_1} < 1, \frac{x_1}{x_1+x_2} < 1]$

$x_3 = y_1 (1 - y_2)$

Clearly, 0< y_1 < ∞ and 0< y_2, y_3< 1

The Jacobian is J= $\begin{matrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{matrix}$

$$= \begin{vmatrix} y_2\, y_3 & y_1 y_3 & y_1\, y_2 \\ y_2(1-y_3) & y_{1(1-y_3)} & -y_1\, y_2 \\ 1-\ y_2 & -y_1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ y_2(1-y_3) & y_1(1-y_3) & -y_1\, y_2 \\ 1-y_2 & -y_1 & 0 \end{vmatrix}$$

= - ${y_1}^2 y_2$

The PDF of $(Y_{1,}\, Y_2, Y_3)$is

$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3)$=$\begin{cases} e^{-y_1}\,|\ {y_1}^2 y_2|\ ,if\ 0 <\ y_1 <\ \infty\ and\ 0 < y_2\ , y_3 < 1 \\ 0\quad , \qquad ow \end{cases}$

$= \begin{cases} \frac{e^{-y_1}\ .{y_1}^{3-1}}{\Gamma(3)} .\, 2y_2\ .1\ , if\ 0 <\ y_1 <\ \infty\ , 0 <\ y_{2,}\ y_3 < 1 \\ 0\quad , \qquad ow \end{cases}$

Where, $f_{Y_1}(y_1) = \begin{cases} \frac{e^{-y_1}\ .{y_1}^{3-1}}{\Gamma(3)}\ , 0 < y_1 <\ \infty \quad \therefore Y_1 \sim Gamma(3) \\ 0\quad , \qquad o\ w \end{cases}$

$$f_{Y_2}(y_2) = \begin{cases} \frac{{y_{2,}}^{2-1}(1-\ y_{2,})^{1-1}}{\beta(2,1)}\ , 0 < y_{2<1}\ \therefore\ Y_2 \sim\ \beta(2,1) \\ 0\quad , \qquad ow \end{cases}$$

And $f_{Y_3}(y_3)$ = $\begin{cases} 1\ \ , 0 < y_3 < 1 \quad , \therefore\ \ Y_3\ \sim U(0,1) \\ 0\quad ,\quad ow \end{cases}$

[Due to independence]

Q51. let $\boldsymbol{f_{X,Y,Z}(x,y,z)}$= $\begin{cases} \frac{\mathbf{6}}{(\mathbf{1}+x+y+x)^{\mathbf{4}}}\ , \boldsymbol{if\ x,y,z} > 0 \\ \mathbf{0}\ \ ,\ \ \boldsymbol{ow} \end{cases}$

Be the PDF of (X, Y, Z). find the distn of U = X+ Y+ Z.

Solution :- U = X+ Y+ Z

V= $\frac{X+Y}{X+Y+Z}$ $\quad here, 0 < U < \infty\ \ and\ 0 < V\ , w < 1$

W= $\frac{X}{X+Y}$

J= - $u^2 v$

$f_{U,V,W}(u,v,w)$= $\begin{cases} \frac{6}{(1+u)^4}\ .|-u^2 v|, 0 < u <\ \infty\ and\ 0 < v, w < 1 \\ 0\quad , \qquad ow \end{cases}$

The PDF of U is , ___________

$$f_U(u)=\begin{cases}\int_0^1\left(\int_0^1 \frac{6u^2v}{(1+u)^4}dv\right)dw\ ,\ if\ 0<u<\infty\\ 0 \quad , \quad ow\end{cases}$$

$$=\begin{cases}\frac{1}{\beta(3,1)}\cdot\frac{u^{3-1}}{(1+u)^{3+1}}\ ,0<u<\infty\\ 0\ ,\ ow\end{cases}$$

Hence U ~ second kind Beta (3, 1)

Q. If X_1, X_2, X_3 ~ N(0, 1). Find the distn s of

$Y_1=\frac{X_1+X_2+X_3}{\sqrt{3}}$

$Y_2=\frac{X_1-X_2}{\sqrt{2}}$

$Y_3=\frac{X_1+X_2-2X_3}{\sqrt{6}}$

Solution:-$f_{X_1,X_2,X_3}(x_1,x_2,x_3)=(\frac{1}{2\pi})^{3/2}\ .\ e^{-\frac{1}{2}(x_1{}^2+x_2{}^2+x_3{}^2)}$; xi ∈ $\mathbb{R}$

Note that, y = $\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&0\\\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{6}}&-\frac{2}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

= a x , wher A is orthogonal, i. e. $AA^T=I_3$.

∴ x= $A^{-1}y=A^Ty$ and

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{3}}&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{3}}&0&-\frac{2}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$$

$\Rightarrow x_1=\frac{y_1}{\sqrt{3}}+\frac{y_2}{\sqrt{2}}+\frac{y_3}{\sqrt{6}}$

$x_2=\frac{y_1}{\sqrt{3}}-\frac{y_2}{\sqrt{2}}+\frac{y_3}{\sqrt{6}}$

$x_3=\frac{y_1}{\sqrt{3}}$ +0.y - $\frac{2y_3}{\sqrt{6}}$

Jacobian = $|\frac{\partial(old\ variable\ )}{\partial(old\ variable\ )}|=|\frac{\partial x}{\partial y}|$or $|\frac{\partial(x_1,x_2,x_3)}{\partial(y_1,y_2,y_3)}|$

$$= \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} = |A^T| = \pm 1.$$

Note that, y′y = x′ A′Ax = x′x $\Rightarrow \sum_{i=1}^{3} y_i^2 = \sum_{i=1}^{3} x_i^2$

Clearly, $y_i \in \mathbb{R}$, i=1, 2, 3

The PDF of Y_1, Y_2, Y_3 is

$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = (\frac{1}{2\pi})^{3/2} . e^{-\frac{1}{2}\sum_{i=1}^{3} y_i^2} .| \pm 1|$, $y_i \in \mathbb{R}$

$= \prod_{i=1}^{3} \{\frac{1}{\sqrt{2\pi}} . e^{-\frac{1}{2} y_i^2}\} = \prod_{i=1}^{3} f_{Y_i}(y_i)$

Hence, $Y_i \sim$ N (0, 1) , i= 1, 2, 3.

Q52. Let $X_1, X_2, \ldots, X_n$ be a r. s. from N (μ, μ), μ > 0.

(a) Find a consistent estimator of μ^2. Is it unbiased?

(b) Find out an UE which is consistent?

Solution :- (a) x̄ ~ N (μ, $\frac{\mu}{n}$)

$\Rightarrow$ E (x̄)= μ

V(x̄)= $\frac{\mu}{n} \longrightarrow 0$ as n $\longrightarrow \infty$

Hence x̄ is consistent for μ.

By invariance property, $\bar{x}^2$ is consistent for μ^2.

But, E ($\bar{x}^2$)= v(x̄)+ E^2(x̄)

$= \frac{\mu}{n} + \mu^2 \neq \mu^2$ [$\because X_i \sim$ N (μ, μ)]

i.e. $\bar{x}^2$ is biased for μ^2.

(b) In a normal sample, x̄ and S^2 are independently distributed.

Also, E (x̄)= μ and E (S^2)= μ

Hence,E(x̄. S^2) = E(x̄) .E (S^2), due to independence.

$= \mu^2$

And var $(\bar{x}.S^2)$= E $(E^2)^2$-$E^2(\bar{x}.S^2)$

= E $(\bar{x}^2.S^4)$ - μ^4

= E $(\bar{x}^2)$. E$(.S^4)$ - μ^4

= {v($\bar{x}$) +$E^2(\bar{x})$ } {v(S^2)+ $E^2(S^2)$ }-μ^4

= $\{\frac{\mu}{n} + \mu^2\}$ $\{\frac{2\mu^2}{n-1} + \mu^2\}$- μ^4

$\longrightarrow 0$ as n $\longrightarrow \infty$

Hence, $\bar{x}.S^2$ is consistent as well as unbiased for μ^2.

Q52. give an example of an estimator which is

(i) Consistent but not unbiased,

(ii) Unbiased but not consistent,

(iii) Consistent as well as unbiased.

ANs :- (i) Let $T_1 = \bar{x} + \frac{1}{n}$

Clearly, $T_1 = \bar{x} + \frac{1}{n}$ is consistent but E (T_1)= $\mu + \frac{1}{n} \neq \mu$

So, it is not unbiased.

[if $\{T_n\}$ is consistent for θ, the $\{T_n + a_n\}$ is consistent for θ if $\lim_{n \to \infty} a_n = 0$.]

(ii) note that, T = $\frac{X_1+X_n}{2}$ is an unbiased estimator of μ.

T ~ n(μ, ${\sigma^2}/{2}$)

Now, P [|T- μ|< ∈] = P $[|\frac{T-\mu}{\frac{\sigma}{\sqrt{2}}}| < \frac{\epsilon\sqrt{2}}{\sigma}]$

= 2 I$[\frac{\epsilon\sqrt{2}}{\sigma}]$-1

$\nrightarrow 1$ as n $\longrightarrow \infty$

Hence, T is unbiased but not consistent for μ.

(iii) Let $X_1, X_2, \ldots, X_n$ be a r. s. from N (μ, σ^2)

Then x̄ ~ n (μ, ${\sigma^2}/{n}$)

E (x̄)= μ , v(x̄) = ${\sigma^2}/{n}$ ⟶ 0 as n⟶ ∞

⇒ x̄ is consistent as well as unbiased.

Q53. Show that for a r. s. from Cauchy distribution with location parameter μ, i.e. C(μ, 1), the sample mean is not consistent for μ but the sample median is consistent for μ.

ANS: - let $X_1, X_2, \ldots, X_n$ be a r. s. from C (μ, 1)

Then x̄ ~ c (μ, 1)

Now, P [|x̄ -μ|< ∈]= P [μ- ∈ <x̄ < μ+ ∈]

$=\int_{\mu-\epsilon}^{\mu+\epsilon} \frac{d\bar{x}}{\pi\{1+(\bar{x}-\mu)^2\}}$

$= [\frac{1}{\pi} \tan^{-1}(\bar{x}-\mu)]_{\mu-\epsilon}^{\mu+\epsilon}$

$=\frac{2}{\pi} \tan^{-1} \epsilon \nrightarrow 1$ as n ⟶∞

Hence x̄ is not consistent for μ.

It can be shown that for large samples,

$\xi_p \sim N\left(\xi_p, \frac{P(1-P)}{n.f^2(\xi_p)}\right)$,

Where, f(.)is the PDF of the distribution.

For, C (μ, 1) distribution, $\xi_p \sim N\left(\xi_p, \frac{1}{4nf^2(\mu)}\right)$

⇒ x̄ ~ N $\left(\mu, \frac{\pi^2}{4n}\right)$ [∵ f(μ)=$\frac{1}{\pi}$]

Hence, for large n, E (x̄)=μ

V (x̄)= $\frac{\pi^2}{4n}$ ⟶ 0 as n ⟶∞

$\Rightarrow$ x̄$(\xi_{\frac{1}{2}})$ is consistent for μ.

Remarks: - By khinchinte's WLLN: $\bar{x} \xrightarrow{P} \mu$, provided E($X_1$)=$\mu$, the population mean exists. In Cauchy population, the population mean does not exist and μ is not the populations mean but it is the population median. Hence for μ, $\bar{X}$ is not consistent, but x̄ is consistent.

Q54. let $X_1, X_2, \ldots, X_n$ be a r. s. from the population with PDF

f (x; θ) = $\begin{cases} e^{-(x-\theta)}, \text{if } x > \theta \\ 0 \ , ow \end{cases}$

Show that $X_{(1)}$ is consistent for θ.

ANS :- $f_{X_{(1)}(x)}$= n[1 $- \int_0^x e^{-(x-\theta)}dx]^{n-1}. e^{-(x-\theta)}$; x> θ

= n $[1 + e^{-(x-\theta)} - 1]^{n-1}. e^{-(x-\theta)}$

= n$e^{-n(x-\theta)}$; x> θ

P $[|X_{(1)} - \theta| < \epsilon]$= P $[\theta < X_{(1)} < \theta + \epsilon]$ = n$\int_\theta^{\theta+\epsilon} e^{-n(x-\theta)}dx$

= n$e^{n\theta} [\frac{e^{-nx}}{-n}]_\theta^{\theta+\epsilon}$

= 1- $e^{-n\epsilon}$

$\longrightarrow$1 as n $\longrightarrow \infty$

$\therefore X_{(1)}$ is consistent for θ.

Q55. If $X_1, X_2, \ldots, X_n$ be a r. s. from f(x) = $\frac{1}{2}$ (1+θx);

-1< x<1, -1<θ <1. Find a consistent estimator of θ.

Solution :- f(x)= $\frac{1}{2}$ (1+θx)I -1< x< 1

$\therefore$ E(X) = $\frac{1}{2}\int_{-1}^{1}(1+\theta x)x dx = \frac{\theta}{3}$

Now, E ($\bar{X}$) = $\frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{\theta}{3}$

$\Rightarrow$ E (3$\bar{X}$) = θ

Now, E (X^2) = $\frac{1}{2}\int_{-1}^{1} x^2(1+\theta x)dx = \frac{1}{2}\int_{-1}^{1}(x^2+\theta x^3)dx = \frac{1}{3}$

$\therefore$ V(X) = E (X^2) -$E^2(X)$

$\Rightarrow$ V(X) = $\frac{1}{3}$ - $\frac{\theta^2}{9}$

V $(\bar{X})$ = $\frac{1}{n^2}.n(\frac{1}{3} - \frac{\theta^2}{9})$= $\frac{1}{n}(\frac{1}{3} - \frac{\theta^2}{9})$

$\therefore$ $\text{lt}_{n\to\infty} V(3\bar{X})$ = $9\text{lt}_{n\to\infty} V(\bar{X})$= $9\lim_{n\to\infty} \frac{1}{n}(\frac{1}{3} - \frac{\theta^2}{9})$=0

$\therefore$ $3\bar{X}$ is a consistent estimator of θ.

Q56. Examine whether the WLLN holds for the following sequences $\{X_n\}$ of independent R.Vs:

I> P $[X_n = -2^n]$ = 2^{-2n-1} = P$[X_n = 2^n]$

P $[X_n = 0]$= 1- 2^{-2n}

II> P $[X_n = -\frac{1}{n}]$= ½ =P $[X_n = \frac{1}{n}]$

Solution :- i> μk= E(X_k)= $(-2^k).\, 2^{-2k-1} + (2^k).\, 2^{-2k-1} + 0.\,(1 - 2^{-2k})$

 = 0

And var(X_k)=$\boldsymbol{\sigma}k^2 = E(\text{X}k^2)$

= $(-2^k)^2 \,.\, 2^{-2k-1} + (2^k)^2 .\, 2^{-2k-1} + 0$

= 1 , k $\in\mathbb{N}$

Now, $\frac{1}{n^2}\sum_{k=1}^{n} \sigma k^2 = \frac{1}{n^2}\sum_{k=1}^{n} 1 = \frac{1}{n} \longrightarrow 0$ as n $\longrightarrow\infty$

Hence, $\{X_n\}$ obeys WLLN , by chebyshev's WLLN.

II) Here μk = 0 and $\boldsymbol{\sigma}k^2 = V(X_k) = E\ (X^2k) = \frac{1}{k^2}$, n $\in\mathbb{N}$

Now, $\frac{1}{n^2}\ \sum_{k=1}^{n} \sigma k^2 = \frac{1}{n^2}\ \sum_{k=1}^{n} \frac{1}{k^2} < \frac{C}{n^2}$

[$\sum_{k=1}^{n} \frac{1}{k^2}$ is a convergent p-series,

$\Rightarrow \sum_{k=1}^{n} \frac{1}{k^2} < \ \sum_{k=1}^{\infty} \frac{1}{k^2} = c$, a finite quantity]

Hence, $\{X_n\}$obeys WLLN, by chebyshev's WLLN.

Q57. Let P [$X_n = -n^P$]= $\frac{1}{2}$ = $P[X_n = n^P]$

Show that WLLN holds for the sequence {X_n} of independent R.V.'s if P < $\frac{1}{2}$

Solution :- here, $\mu_k = E(X_k) = 0$

σk^2= V (X_k) $= E(X_k)^2 = (-k^P)^2.\frac{1}{2} + (k^P)^2.\frac{1}{2}$

= k^{2P}, k ∈ℕ

Now, $\frac{1}{n^2}\sum_{k=1}^{n}\sigma k^2 = \frac{1}{n^2}\sum_{k=1}^{n}k^{2P} < \frac{1}{n^2}\int_1^n x^{2P}dx$

= $\frac{n^{2p+1}-1}{n^2(2P+1)}$

Now, $0 \leq \frac{1}{n^2}\sum_{k=1}^{n}\sigma k^2 < \frac{n^{2p+1}-1}{n^2(2P+1)} < \frac{n^{2p-1}}{(2P+1)}$ ⟶0 as n ⟶∞ [if 2P-1< 0, if p < $\frac{1}{2}$]

⇒ if p < $\frac{1}{2}$, $\frac{1}{n^2}\sum_{k=1}^{n}\sigma k^2$ ⟶0 as n ⟶∞

Hence, {X_n} obeys WLLN if p < $\frac{1}{2}$.

Q58. Decide whether WLLN holds for the sequence,{ X_n} o0f independent R.V.'s :

P [$X_n = \pm 2^{-n}$]= $\frac{1}{2}$

⇔ P [$X_n = -2^{-n}$]= $\frac{1}{2}$= P[$X_n = 2^{-n}$]

Solution :- Here, μk=0,

And σk^2 = V (X_k) $= E\left({X_k}^2\right) = 2^{-2k}, k \in \mathbb{N}$

Now, $\frac{1}{n^2}\sum_{k=1}^{n}\sigma k^2 = \frac{1}{n^2}\sum_{k=1}^{n}2^{-2k} = \frac{1}{n^2}.\frac{\frac{1}{4}\{1-(\frac{1}{4})^n\}}{1-\frac{1}{4}}$

= $\frac{1}{3}.\frac{1}{n^2}\{1-\left(\frac{1}{4}\right)^n\} < \frac{1}{3n^2}$ ⟶0 as n ⟶∞

Hence, $\lim_{n\to k}\frac{1}{n^2}\sum_{k=1}^{n}\sigma k^2 = 0$

⇒ X_n} obeys WLLN by chebyshev's WLLN.

II) μk=0

And $\sigma k^2 = E\left({X_k}^2\right) = (-k)^2 . \frac{1}{2\sqrt{k}} + (k)^2 . \frac{1}{2\sqrt{k}} + 0$

$= k^{3/2}$

Now, $\frac{1}{n^2} \sum_{k=1}^{n} \sigma k^2 = \frac{1}{n^2} \sum_{k=1}^{n} k^{3/2}$

For, large n, $\frac{1}{n} \sum_{k=1}^{n} (\frac{k}{n})^{3/2} \simeq \int_0^1 x^{3/2} dx = \frac{2}{5}$

$\Rightarrow \sum k^{3/2} \simeq \frac{2n^{5/2}}{5} = \frac{2}{5}\sqrt{n} \nrightarrow 0 \longrightarrow \infty$ as n $\longrightarrow \infty$

As $\frac{1}{n^2} \sum_{k=1}^{n} \sigma k^2 \nrightarrow 0$ as n $\longrightarrow \infty$

We cannot draw any conclusion by chebyshev's WLLN, whether WLLN holds or not.

Q59. Let (X_1, X_2, X_3) be a r.s. from Bin(1, p). Is T= $X_1 + 2X_2 + X_3$ sufficient for p ? is $X_1 + X_2 + X_3$ is sufficient for p?

ANS :- (i) Here T takes the value 0, 1, 2, 3, 4.

P $[X_1 = 1, X_2 = 0, X_3 = 1 | T = 2\ ]$

$= \frac{P[X_1=1, X_2=0, X_3=1\ ; T=2]}{P[T=2]}$

$= \frac{P[X_1=1, X_2=0, X_3=1\ ]}{P[X_1=1, X_2=0, X_3=1] + P[P[X_1=0, X_2=1, X_3=0]}$

$= \frac{p^2(1-p)}{p^2(1-p) + p(1-p)^2} = \frac{p}{p+1-p}$ = p, which depends on p.

Hence T is not sufficient for p

(ii) Here, $X_1 + X_2 + X_3 = T$

Let us consider a specific case, $X_1 = 1, X_2 = 1, X_3 = 0$ and T =1

Here, $X_1 + X_2 + X_3 = 1$ for,

$\{(X_1 = 1, X_2 = 1, X_3 = 0), (X_1 = 1, X_2 = 0, X_3 = 1), (X_1 = 0, X_2 = 1, X_3 = 1), (X_1 = 0, X_2 = 0, X_3 = 1)\ \}$

$\therefore$ P $[X_1 = 1, X_2 = 1, X_3 = 0 | T = 1]$

$= \begin{cases} \frac{\text{P}\ [X_1=1, X_2=1, X_3=0]}{P[T=1]}, & \text{if T} = 1 \\ 0, & ow \end{cases}$

$$= \begin{cases} \frac{p^2(1-p)}{3p^2(1-p)+(1-p)^2p}, & if\ T=1 \\ 0 & , ow \end{cases}$$

$$= \begin{cases} \frac{P}{2P+1}, & if\ T=1 \\ 0, & ow \end{cases}$$

E. T is not sufficient for p.

Q60. Let $X_1, X_2, \ldots, X_n$ be a r. s. from the following PDF & find the non trival sufficient statistic in each case :

(i) f(x ; θ)= $\begin{cases} \theta x^{\theta-1}; & 0 < x < 1 \\ 0, & ow \end{cases}$

(ii) f(x ; μ)= $\frac{1}{|\mu|\sqrt{2\pi}} . e^{-\frac{(x-\mu)}{2\mu^2}}$ **; x $\in \mathbb{R}$**

(III) f (x ; α,β)= $\begin{cases} \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\beta(\alpha,\beta)}, & 0 < x < 1 \\ 0, & ow \end{cases}$

(iv) f(x ; μ, λ)= $\begin{cases} \frac{1}{\lambda} e^{-\frac{(x-\mu)}{\lambda}}, & if\ x > \mu \\ 0, & ow \end{cases}$

(v) f(x; μ, σ)= $\begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(|nx-\mu)^2}, & if\ x > 0 \\ 0, & ow \end{cases}$

(vi) f(x;α,θ)= $\begin{cases} \frac{\theta\, \alpha^\theta}{x^{\theta+1}} & if x > \alpha \\ 0, & ow \end{cases}$

(vii) f(x; θ) = $\begin{cases} \frac{2(\theta-x)}{\theta^2}; & 0 < x < \theta \\ 0, & ow \end{cases}$

Ans:- (I) The joint PDF of $X_1, X_2, \ldots, X_n$ is

f(x) = $\theta^n (\prod_{i=1}^n x_i)^{\theta-1}$

= $g_{\theta\ \{\prod_{i=1}^n x_i\}}$. h(x). where h(x)=1

And T (x)= $\prod_{i=1}^n x_i$

∴ By Neymon – Fisher factorization criterion,

T = $\prod_{i=1}^n x_i$ is sufficient for θ.

(ii) f(x; μ, **σ**)= $\frac{1}{|\mu|\sqrt{2\sigma}} . e^{-\frac{(x-\mu)}{2\sigma^2}}$

So, X ~ N (μ, μ^2), where $\mu \neq 0$.

By Ex.(3). T (x)= $(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} {X_i}^2)$ is sufficient for μ.

Note : - If in the range of X_i , there is the parameter of the distribution present then we have to use the concept of indicator function($X_{(1)}$ or $X_{(n)}$) or $\min_i\{\chi_i\}$.

(iii> $f_\theta(x) = \frac{1}{B(\alpha,\beta)}\ x\,.(1-x)$, if 0< x< 1 , α, β> 0

∴ Joint PDF of $X_1, \ldots, X_n$ is

f(x̰)= $\left[\frac{1}{B(\alpha,\beta)}\right]^n (\prod_{i=1}^{n} x_i)^{\alpha-1}(\prod_{i=1}^{n} 1-x_i)^{\beta-1}$

= g(T((x̰) ; α, β)h(x̰), where, h(x̰)= 1 and T (x̰) =$(\prod_{i=1}^{n} x_i\ , \prod_{i=1}^{n} 1-x_i)$ is jointly sufficient for (α, β)

(iv> f(x̰)= $\frac{1}{\theta^n}\ . e^{-\sum_{i=1}^{n}\frac{(x_i-\mu)}{\sigma}}$ if x_i > μ

= $\frac{1}{\sigma^n}$. exp{$\frac{-\sum_{i=1}^{n} x_i - n\mu}{\sigma}$}. I ($x_{(1)\ ,\mu}$) where I (a, b)= 1 if a ≥ b

= 0 ow

= g($\sum_{i=1}^{n} X_i, x_{(1)\ ;\sigma,\mu}$) . h(x̰), where h(x̰)= 1.

Thus, $X_{(1)}$ and $\sum_{i=1}^{n} X_i$, are jointly sufficient statistic for μ and **σ**.

<v> f(x; μ, **σ**)= $\frac{1}{x\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2\sigma^2}(|nx-\mu)^2}$, $if\ x > 0$

The joint PDF of x̰ is

F(x̰) = $\frac{1}{(\prod_{i=1}^{n} x_i)\sigma^n(\sqrt{2\pi})^n}$. exp{$-\frac{1}{2\sigma^2}\ \sum_{i=1}^{n}(|nx_i-\mu)^2$,} if x_i > 0

= $\frac{1}{\sigma^n(\sqrt{2\pi})^n}\ . e^{-(\frac{\sum(1nx_i)^2}{2\sigma^2} - \frac{\mu\sum 1nx_i}{\sigma^2} + \frac{n\mu^2}{\sigma^2})} . \frac{1}{(\prod_{l=1}^{n} x_i)}$

= T ($\sum_{i=1}^{n} |nx_i, \sum_{i=1}^{n}(|nx_i)^2\ ;\ \mu, \sigma$). h(x̰) ; where,

h(x̰) = $\frac{1}{\prod_{i=1}^{n} x_i}$; T (x̰)= $\sum_{i=1}^{n} |nx_i, \sum_{i=1}^{n}(|nx_i)^2$

is sufficient for μ and **σ**.

(vi) f (x̰)= $\theta^n\ \frac{(\alpha^\theta)^n}{\prod_{i=1}^{n}({x_i}^{\theta+1})}$ if x_i > α

$= (\theta\alpha^{\theta})^n . \frac{1}{\prod_{i=1}^{n}\{x_i\}^{\theta+1}}\ I(x_{(1),\alpha})$ if $x_{(1)>\alpha}$ where I (a, b)=1 if a > b

= 0 ow

= g ($\prod_{i=1}^{n} x_i$, $x_{(1);\ \alpha,\ \alpha}$). h (x ̰); where , h (x ̰)= 1 and hence

T =$\prod_{i=1}^{n} x_i$, $x_{(1)}$ is sufficient for θ and α.

(vii> f (x ̰) $= \frac{2^n}{\theta^{2n}} \prod_{i=1}^{n}(\theta - x_i)$; $0 < x_i < \theta$

$= \left(\frac{2^n}{\theta^{2n}}\right)^n . (\theta - x_1)(\theta - x_2) \dots . (\theta - x_n)$; $0 < x_i < \theta$

These cannot be expressed in the form of factorization criterion.

So $(X_1, X_2, \dots, X_n)$ or$(X_{(1)}, X_{(2)}, \dots, X_{(n)})$ are trivally sufficient for here , ̰.there is no non- trival sufficient statistic.

Q61. Let $X_1, \dots, X_n$ be a r.s. from gamma distn with pdf

$$f_{\theta}(x) = \frac{\alpha^P}{\Gamma(P)} \exp[-\alpha x] x^{P-1}$$ **if 0< x< ∞ , where α> 0, P > 0**

Show that $\sum_i X_{i\ and}\ \prod_i X_i$ are jointly sufficient for (α, P)

Solution:- f(x ̰)= $\{\frac{\alpha^P}{\Gamma(P)}\}^n . \exp[-\alpha \sum_i X_i] . (\prod_i X_i)^{P-1}$

= g (T(x ̰); , P). h (x ̰); where h(x ̰)=1

∴ T (x ̰)= ($\sum_{i=1}^{n} X_i, \prod_{i=1}^{n} X_i$) is jointly sufficient for (α, P).

Q62. If f(x)= $\frac{1}{\theta}\ e^{-x/\theta}$; $0 < x < \theta. Find\ a\ sufficient\ estimator\ for\ \theta.$

Solution:- f(x ̰)= $\frac{1}{\theta^n}\ . exp\ \{-\frac{1}{\theta}\sum_{i=1}^{n} x_i\}$

= g $\{\sum_{i=1}^{n} x_i , \theta\}$. h (x ̰); where h (x ̰)= 1.

∴T = $\sum_{i=1}^{n} x_i$ is sufficient statistic for θ.

Q63. If $f_{\theta}(x)$= ½ ; θ -1< x < θ -1 , then show that $X_{(1)}$ and $X_{(n)}$ are jointly sufficient for θ. ($X_i \sim U(\theta - 1, \theta + 1)$)

Solution:- f(x ̰)=$(\frac{1}{2})^n$

$= \frac{1}{2^n} . I\left(\theta - 1, x_{(1)}\right) I\left(x_{(n)},\ \theta + 1\right); \theta - 1 < x_{(1)} < x_{(n)} < \theta + 1$

Where I (a, b)= $\begin{cases} 1 & if\ a < b \\ 0 & if\ a \geq b \end{cases}$

= g (T (x̰); θ)h(x̰); where h(x̰)= $\frac{1}{2^n}$.

∴ T (x̰) = $(X_{(1)}, X_{(n)})$is jointly sufficient for θ.

Q64. let $X_1, X_2, \dots, X_n$ be a R.S. from c (θ, 1), where θ is the location parameter, S. T. there is no sufficient statistic other than the trivial statistic $(X_1, X_2, \dots, X_n)$ or $(X_{(1)}, X_{(2)}, \dots, X_{(n)})$

If a random sample of size n ≥ 2 from a Cauchy distn with p.d.f.

$f_\theta(x)$= $\frac{1}{\pi[1+(x-\theta)^2]}$, where -∞ < θ <∞, is considered.

Then can you have a single sufficient statistic for θ?

Solution:- The PDF of $(X_1, \dots, X_n)$ is

$$\prod_{i=1}^{n} f\ (X_i, \theta) = \frac{1}{\pi^n \{\prod_{i=1}^{n}[1 + ((X_i - \theta)^2]\}}$$

Note that, $\prod_{i=1}^{n}\{1 + ((X_i - \theta)^2\ \}$

= $\{1 + ((x_1 - \theta)^2\}\{1 + ((x_2 - \theta)^2\}\dots.\ \{1 + ((x_n - \theta)^2\}$

= 1+ term involving one X_i+ term involving two $X_i's$ + ……. + term involving all $X_i's$.

= 1+ $\sum_i (x_i - \theta)^2 + \sum_i \sum_{\neq j} (x_i - \theta)^2\ ((x_j - \theta)^2 + \ \dots. + \prod_{i=1}^{n}((x_i - \theta)^2$

Clearly, $\prod_{i=1}^{n} f(x_i, \theta)$cannot be written as g (T (x̰), θ). h(x̰)

For a statistic other than the trivial choices

$(X_1, \dots, X_n)$ or $(X_{(1)}, \dots, X_{(n)})$.

Hence there is no non-trivial sufficient statistic

Therefore, in this case, <u>no reduction in the space is possible</u>.

⇒ The whole set $(X_1, \dots, X_n)$ is jointly sufficient for θ.

Q65. Let X_1 *and* X_2 be iid RVS having the discrete uniform distribution on {1, 2, ……, N}, where n is unknown. Obtain the conditional distribution of X_1 , X_2 given (T = max (X_1 , X_2))

Hence, show that T is sufficient for n but $X_1 + X_2$ is not.

ANS :- (i) p (T= t)= P [max $(X_1, X_2) = t$]

= P [$X_1 < t,\ X_2 = t$]+ P [$X_1 = t,\ X_2 < t$]+ P [$X_1 = t,\ X_2 = t$]

= P [$X_1 < t$] P [$X_2 = t$]+ P [$X_1 = t$] P [$X_2 < t$] + P [$X_1 = t$] P [$X_2 = t$]

Now, P [$X_1 < t$]= P [$X_2 = 1$]+ P [$X_2 = 2$]+ …. + P [$X_2 = t-1$]

$$= \underbrace{\frac{1}{N} + \frac{1}{N} + \ \ldots + \frac{1}{N}}_{(t-1)\ \text{times}}$$

$= \frac{t-1}{N}$ & P [$X_1 = t$]= P [$X_2 = t$]= $\frac{1}{N}$

∴ P [T = t]= $\frac{1}{N}. \frac{t-1}{N} + \frac{t-1}{N}.\frac{1}{N} + \frac{1}{N}.\frac{1}{N}$

$= \frac{2\,(t-1)+1}{N^2}$

∴ P [$X_1 = x_1, X_2 = x_2 \mid T = t$]= $\begin{cases} \frac{P[X_1 = x_1, X_2 = x_2]}{P\,[T=t]}, & if\ Max(x_1, x_2) = t \\ 0, & ow \end{cases}$

$= \frac{\frac{1}{N}.\frac{1}{N}}{\frac{2\,(t-1)+1}{N^2}} = \frac{1}{2\,(t-1)+1}$, which is independent of N.

(ii) T = $X_1 + X_2$ then,

For $2 \le t \le N+1$;P [T=t]= P [$X_1 = 2, X_2 = t-1$]+ P [$X_1 = 2, X_2 = t-2$] + …+P [$X_1 = 2, X_1 = t-1, X_2 = 1$]

$= \frac{(t-1)}{N^2}$

For N +2≤ $t \le 2N$; P [T=t]= P[$X_1 = t-N, X_2 = N$] + P [$X_1 = t-N+1, X_2 = N-1$]+ …..+P [$X_1 = N, X_2 = t-N$]

$= \frac{2N-t+1}{N^2}$

∴ P [$X_1 = x_1\ ;\ X_2 = x_2 \mid T = t$]= $\frac{P\,[X_1 = x_1\ ; X_2 = x_2\ ]}{P\,[X_1 + X_2 = t]}$

$$= \begin{cases} \dfrac{\frac{1}{N^2}}{\frac{(t-1)}{N^2}} = \dfrac{1}{t-1} \quad if\ X_1 + X_2 = t \\ \dfrac{\frac{1}{N^2}}{\frac{2N-t+1}{N^2}} = \dfrac{1}{2N-t+1} \quad if\ X_1 + X_2 = t \end{cases}$$

Which depends on N, so for the 2nd case $(X_1 + X_2)$is not sufficient.

Q66. Let $X_1, X_2, \dots, X_n$ be a R.S. from one of the following two PDFs

If θ = 0, f(x/θ)= $\begin{cases} \mathbf{1} \ , \mathbf{0} < x < 1 \\ \mathbf{0} \ , \quad \boldsymbol{ow} \end{cases}$

If θ=1, , f(x/θ)= $\begin{cases} \frac{\mathbf{1}}{2\sqrt{x}} \ , \mathbf{0} < x < 1 \\ \mathbf{0} \ , \quad \boldsymbol{ow} \end{cases}$

Find the MLE of θ.

Solution: - The Likelihood function is

L (θ/ x̰)= $\prod_{i=1}^{n} f\left(\frac{x_i}{\theta}\right), \theta \in \Omega = (0, 1)$

When θ = 0, L (θ/ x̰) = $\begin{cases} 1 \ \ if \ \ 0 < x_i < 1 \ \forall \ i = 1(1)n \\ \quad 0 \quad , \quad 0w \end{cases}$

When θ= 1, L (θ/ x̰) = $\begin{cases} \dfrac{1}{2^n \sqrt{\prod_{i=1}^{n} x_i}} \ , 0 < x_i < 1 \ , i = 1(1)n \\ \quad 0 \ , \quad ow \end{cases}$

Now, $\frac{L\ (\theta=1/\ x̰)}{L\ (\theta=\frac{0}{x̰})} \gtreqless 1$

Iff $\frac{1}{\sqrt{4^n} G^n} \gtreqless 1$, where G = $(\prod_{i=1}^{n} x_i)^{1/n}$

Iff 4 G $\lesseqgtr$ 1 iff G$\lesseqgtr \frac{1}{4}$

Hence MLE of θ is θ’ = $\begin{cases} 1\ if\ G < \frac{1}{4} \\ 0\ if\ G > \frac{1}{4} \\ 0{,}1\ if\ G = \frac{1}{4} \end{cases}$

Q67. Let $X_1, \dots, X_n$ be a R.S. from N (μ, σ^2), μ ϵR, σ > 0

Find the MLE of (μ, σ^2).

Solution: - Likelihood function:

L (μ, $\sigma^2/\underset{\sim}{x}$) = $\frac{1}{(2\pi\sigma^2)^{n/2}} \cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$; $x_i \in \mathbb{R}$ where $\mu \in R, \sigma > 0$

$\Rightarrow$ |nL (μ, $\sigma^2/\underset{\sim}{x}$)= constant (-$\frac{n}{2}|n\sigma^2 - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$)

$0 = \frac{\partial |nL}{\partial \mu} = -\frac{1}{2\sigma^2}\sum 2(x_i-\mu)(-1) = \frac{\sum x_i}{\sigma^2} - \frac{n\mu}{\sigma^2}$

$0 = \frac{\partial |nL}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{\sum(x_i-\mu)^2}{2\sigma^4}$

$\Rightarrow \begin{cases} \acute{\mu} = \bar{x} \\ \sigma^2 = \frac{1}{n}\sum(x_i-\bar{x})^2 \end{cases}$, the likelihood function has a unique solution.

Note that, the matrix of second order partial derivatives at (μ, σ^2) is

$\begin{pmatrix} \frac{\delta^2 |nL}{\partial \mu^2} & \frac{\delta^2 |nL}{\delta\mu\delta\sigma^2} \\ \frac{\delta^2 |nL}{\delta\sigma^2\delta\mu} & \frac{\delta^2 |nL}{\delta(\sigma^2)^2} \end{pmatrix}$ (μ, σ^2)= (μ, σ^2)

$= \begin{pmatrix} -\frac{n}{\sigma^2} & 0 \\ 0 & -\frac{n}{2\sigma^4} \end{pmatrix}$ is negative definite (n. d.)

Hence, L (μ, $\sigma^2/\underset{\sim}{x}$) is maximum at ($\mu$, σ^2)= (μ, σ^2)

Therefore, the MLE of (μ, σ^2) is

(μ, σ^2) = ($\bar{x}, S^2$) where nS^2= $\sum_{i=1}^{n}(X_i-\bar{x})^2$.

Q68. Let $X_1, \dots, X_n$ be a R.S. from f (x; μ,)= $\frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}$; $x \in R$,where μ ϵR, σ > 0. Find the MLE of μ and σ.

Solution:- The log- likelihood function is

L (μ, $\sigma^2/\underset{\sim}{x}$) = -n|n2- n|n$\sigma$- $\frac{1}{\sigma}\sum|x_i-\mu|$; $\mu \in R, \sigma > 0$

[As$\sum|x_i-\mu|$is not differentiable w.r.t. μ, hence the derivative technique is not applicable for maximizing |nL w.r.t.μ]

We adopt two stage maximization:-

First fix σ, and then maximize |nL for variation in μ.

For fixed σ, |nL is maximum,

Iff, $\sum | x_i - \mu|$is minimum

Iff, μ=x̄= the sample median

= μ, say.

Now, we maximize |nL (μ, $\sigma^2/\underset{\sim}{x}$)=-n|n2-n|n$\sigma$- $\frac{1}{\sigma}\sum | x_i - \mu|, w.r.t.\sigma$

Note that $\frac{\delta}{\delta\sigma}$|nL ($\mu$, $\sigma^2/\underset{\sim}{x}$)

= -$\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum | x_i - \mu|$

= -$\frac{n}{\sigma^2}\{\sigma - \frac{1}{n}\sum | x_i - \mu|\}$

$$\begin{cases} > 0, \sigma < \frac{1}{n}\sum | x_i - \mu| \\ < 0, \sigma > \frac{1}{n}\sum | x_i - \mu| \end{cases}$$

By, 1st derivative test, |nL (μ, $\sigma^2/\underset{\sim}{x}$) is maximum at σ=$\frac{1}{n}\sum_{i=1}^{n} |x_i - \mu|$

Hence, the MLE of μ and σ are μ= x̄ , σ= $\frac{1}{n}\sum | x_i - \bar{x} |$.

Q69. let $X_1, X_2, \dots, X_n$ be a R.S. from

F(x; μ,)= $\begin{cases} \frac{1}{\sigma}\, e^{-(x-\mu)/\sigma}, & if\ x > \mu \\ 0, & ow \end{cases}$

Where $\mu \in$ R, σ > 0. Find the MLE of (i) μ and σ

(ii)μ when σ= μ (>0)

Solution:- (i) The likelihood function is

L (μ, σ/x̄)= $\begin{cases} \frac{1}{\sigma^n} \cdot e^{-\frac{\sum(x_i-\mu)}{\sigma}}; & if\ x_{(1)} \geq \mu \\ 0, & ow \end{cases}$

$\mu \in$ R, σ> 0

We adopt two stage maximization.

First fix σ, then maximize L (μ, σ/x̄) w.r.t.μ

For fixed σ, L (μ, σ/x̄) is maximum

Iff $\sum(x_i - \mu)$ is minimum subject to $\mu \leq x_{(1)}$

Iff μ is as large as possible subject to the restriction $\leq x_{(1)}$.

Iff μ= $x_{(1)}$= μ (say)

Now we shall maximize L (μ, σ/x̄) w.r.t.σ

Now, in L(μ, σ/x̄)= -nlnσ-$\frac{\sum(x_i-\mu)}{\sigma}$

Note that, $\frac{\delta}{\delta\sigma}\ ln$L(μ,$\frac{\sigma}{\bar{x}}$)= -$\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum(x_i - \mu)$

= $\frac{n}{\sigma^2}\{\sigma\text{-}(\bar{x} - x_{(1)})\}$

$$\begin{cases} > 0\ if\ \sigma < \bar{x} - x_{(1)} \\ < 0\ if\ \sigma > \bar{x} - x_{(1)} \end{cases}$$

Hence, L(μ, σ/x̄) is maximum at σ=$\bar{x} - x_{(1)}$= σ

Therefore, The MLE of μ and σ are μ= $x_{(1)}$ and σ= $\bar{x} - x_{(1)}$

(ii) When σ= μ> 0

L (μ/ $\underset{\sim}{x}$) = $\begin{cases} \frac{1}{\mu^n}\ e^{-\frac{\sum(x_i-\mu)}{\mu}}; x_{(1)} \geq \mu \\ 0 \qquad\qquad , ow \end{cases}$

L (μ/ $\underset{\sim}{x}$) is maximum iff

For $\mu \leq x_{(1)}$

$$\frac{\delta}{\delta u}\ ln\text{L} = \frac{\delta}{\delta u}\ \{-nln\mu - \frac{1}{\mu}\sum(x_i - \mu)\ \}$$

$$= -\frac{n}{\mu^2}\ (\mu - \bar{x})$$

$$\begin{cases} > 0\ if\ \mu < \bar{x} \\ < 0\ if\ \mu > \bar{x} \end{cases}$$

⇒ L (μ/ $\underset{\sim}{x}$) is maximum at $\mu = \bar{x}$

From the graph for $\mu \leq x_{(1),}$ L (μ/ $\underset{\sim}{x}$) is maximum at $\mu = x_{(1),}$

Therefore, $\mu = x_{(1),}$ is the MLE of μ.

Q70. Let X be a single observation from the PDF

F (x; $\boldsymbol{\theta}$) = {$\frac{\mathbf{1}}{\boldsymbol{\pi}\{\mathbf{1}+(\boldsymbol{x}-\boldsymbol{\theta})^{\mathbf{2}}\}}$, $\boldsymbol{x} \in \mathbb{R}$

Show that the test

$$\boldsymbol{\Phi}\ (\mathbf{x}) = \begin{cases} \mathbf{1}, \frac{\boldsymbol{f}(\boldsymbol{x},\mathbf{1})}{\boldsymbol{f}(\boldsymbol{x},\mathbf{0})} > \boldsymbol{k} \\ \mathbf{0} \quad , \boldsymbol{ow} \end{cases}$$

Is an MP test of $\boldsymbol{H_0}: \boldsymbol{\theta} = \mathbf{0}$ aginst $\boldsymbol{H_1}: \boldsymbol{\theta} = \mathbf{1}$ of its size.

Solution:- For a particular value of k, the test

$$\Phi\ (\mathrm{x}) = \begin{cases} 1, \frac{f(x,1)}{f(x,0)} > k \\ 0 \quad , ow \end{cases}$$

Is an MP test of $H_0: \theta = 0$ aginst $H_1: \theta = 1$ of its size, by NP lemma,

Now, $\frac{f(x,1)}{f(x,0)} > k \Rightarrow \frac{1+x^2}{1+(x-1)^2} > k$

$\Rightarrow x^2(k-1) - 2kx + (2k-1) < 0$

[If (k-1)> 0, $x^2 - \frac{2k}{(k-1)}x + \frac{2k-1}{k-1} < 0$

⇒ (x- α)(x-β)< 0

Where, $\alpha+\beta = \frac{2k}{(k-1)}$, and $\alpha\beta = \frac{2k-1}{k-1}$

⇒ $\alpha <$ x $< \beta$

In the given MP test α=1, β=3

Hence, 1+3 =$\frac{2k}{(k-1)}$ ⇒ k=2]

Set, k=2, $\frac{f(x,1)}{f(x,0)} > 2$

$\Rightarrow$ 1< x < 3

For k=2, the test Φ (x)=$\begin{cases} 1 & ,1 < x < 3 \\ 0 & ,ow \end{cases}$

Is an MP test of H_0 against H_1 of its size

= E [Φ (x)/ H_0]= P [1< x<3/θ=0]

= $\int_1^3 \frac{1}{\pi(1+x^2)} dx$= $\frac{1}{\pi}$ [$\tan^{-1} x$]3,1

=$\frac{1}{\pi}$ [$\tan^{-1} 3 - \tan^{-1} 1$]

= $\frac{1}{\pi} \tan^{-1} \frac{3-1}{1+3.1}$

= $\frac{1}{\pi} \tan^{-1} \frac{1}{2}$.

Q71. Find an MP test of testing H_0 such that $H_0 : X \sim f_0(x)$ against $H_1 : X \sim f_1(x)$ of its size, where

$\boldsymbol{f_0(x)= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, x \in R}$

$\boldsymbol{f_1(x)= \frac{1}{2} e^{-|x|}, x \in R}$

S.T. the power of the test is greater than its size.

Solution:- By N-P lemma, for a particular value of k, the test

Φ(x)=$\begin{cases} 1 & ,\frac{f_1(x)}{f_0(x)} > k \\ 0 & ,ow \end{cases}$

Is an MP test of H_0 against H_1 of its size.

Now, $\frac{f_1(x)}{f_0(x)} > k$

$\Rightarrow e^{\frac{1}{2}\{x^2-2|x|\}} > k_1$

$\Rightarrow e^{\frac{1}{2}}\{(|x|-1)^2 - 1\} > k_1$

$\Rightarrow (|x|-1)^2 > {k_2}^2, k_2 > 0$

$\Rightarrow$ |x|-1 < -k_2 or |x|-1>k_2

$\Rightarrow$ |x|< C_1 or |x|> C_2

[Alternative: - note that $f_1(x)$ has more probability in its tails and near 0 than $f_0(x)$ has. If either a very large or very small value of x is observed, we suspect that H_1 is true rather thanH_0. For some C_1 andC_2, we shall reject H_0 iff $\frac{f_1(x)}{f_0(x)} > k$

To |x|< C_1 or |x|<C_2.]

Hence, for some C_1 andC_2, the test

Φ(x) = $\begin{cases} 1\text{ , |x|} < C_1 \text{ or |x|} < C_2 \\ \quad 0\text{ , } ow \end{cases}$

Is an MP test of H_0 against H_1 of its size

Note, that, $\beta_{\Phi(}f_1) = P[1 \times 1 < C_1 or\ 1 \times 1 < C_2\ ]$

= $\int_w^{f_1} f_1(x)dx\ , w = \{x: |x| < C_1\ or\ |x| >\ C_2\ \}$

>$\int_w\ f_0\ (x)dx$, as $f_1(x) >\ f_0(x) \forall x \epsilon w$

= $P_{f_0}[1 \times 1 < C_1\ or\ 1 \times 1 < C_2\ ]$

= $\beta_\Phi(f_0)$. (Proved).

Q72. Find an MP test of H_0 : X~N(0, ½) against H_1 : X~c (0, 1) of its size .

Solution :- For a given K , the test Φ (x) = $\begin{cases} 1, \frac{f_1(x)}{f_0(x)} > k \\ \quad 0, ow \end{cases}$

Is an MP test of H_0 against H_1 of its size,

By N-P lemma,

Note that , $\frac{f_1(x)}{f_0(x)} > k$

$\Rightarrow\ \frac{e^{x^2}}{1+x^2}>\ k_1$, say

Let u (x) = $\frac{e^{x^2}}{1+x^2}$

Now, u′ (x)= $\frac{(1+x^2)e^{x^2}.2x - e^{x^2}.2x}{(1+x^2)^2}$

$= \frac{2x^3.e^{x^2}}{(1+x^2)^2}$

[u′ (0)=0 ⇒$2x^3.e^{x^2} = 0 \Rightarrow x = 0 \; or e^{x^2} = 0 \Rightarrow x^2 = \infty$]

$=\begin{cases} < 0 , if \; x < 0 \\ > 0 , if \; x > 0 \end{cases}$

From the graph ,u(x) >k_1

⇔ |x|>c_1

Hence,for a particular

Value of c_1, the test

Φ(x) = $\begin{cases} 1 , |x| > C_1 \\ 0 , \; ow \end{cases}$

Is an MP test H_0 against H_1 of its size.

Q73. Find on MP test at level α= 0.05 for testing H_0: X ~ N (0, 1)against H_1: X ~c (0, 1).

Solution:- for a given k, the test Φ(x) = $\begin{cases} 1 \; , \frac{f_1(x)}{f_0(x)} > k \\ 0 \; , ow \end{cases}$

Is an MP test of H_0 against H_1 of its size, by NP lemma. Note that. R (x)= $\frac{f_1(x)}{f_0(x)} > k$

$\Rightarrow \frac{e^{x^2/2}}{1+x^2} > k_1$, say.

Let u (x) =$\frac{e^{x^2/2}}{1+x^2}$

Note that, u′ (x)= $\begin{cases} < 0 , x < -1 \\ > 0 , -1 < x < 0 \\ < 0 \; , 0 < x < 1 \\ > 0 , x > 1 \end{cases}$

[for k> 0.7979, then the critical region:

|x| $> c_2$ with size< 0.1118.

For 0.6524≤ k≤ 0.7979,

Then critical region:

$|x| > c_1$ or $|x| > c_2$ with size ϵ (0.1118, 0.3913)

For, k < 0.6524, the critical region: x ϵR with size=1]

For α= 0.05, a small quantity, then u(x)> k_1, where k_1 is such that P [u(x)> k_1/H_0]=0.05 and from the graph u(x)> $k_1 \Leftrightarrow$ |x|> c_2.

Hence, $\Phi(x) = \begin{cases} 1, |x| > C_2 \\ 0, \; ow \end{cases}$

Is an MP test of H_0 against H_1 at level α= 0.05, where

$$0.05 = P[|x| > c_2/H_0]$$

=P [[$|x| > c_2$/X $\sim$ N (0, 1)]

= 2 [1- Φ (c_2)]

$\Rightarrow$ 1- $\overline{\phi}(c_2)$ = 0.025 = 1-$\overline{\phi}$ (0.025)

$\Rightarrow$ c_2= 0.025

=1.96

Hence, $\Phi(x) = \begin{cases} 1, |x| > 1.96 \\ 0, \; ow \end{cases}$

Is an MP test for testingH_0: X ~ N (0, 1) against H_1: X ~c (0, 1) at level α= 0.05.

Q74. Let $X_1, \dots, X_n$ be a R.S. from f(x; θ) = $\begin{cases} \theta\, e^{-\theta x}, if\ x > \theta \\ 0 \;, ow \end{cases}$

Find the size α LRT of (i) H_0:$\theta = \theta_0$ $against$ H_1: $\theta \neq \theta_0$

(ii) H_0:$\theta = \theta_0$ $against$ H_1: $\theta > \theta_0$

($iii >$ H_0:$\theta \geq \theta_0$ $against$ H_1: $\theta < \theta_0$

Solution:- The likelihood function is

$$L\ (\underset{\sim}{x};\ \theta) = \begin{cases} \theta^n e^{-\theta \sum_{i=1}^{n} x_i}, if\ x_i > \theta \\ 0 \;\;, ow \end{cases}$$

Where , θ> 0

(i) To test H_0:$\theta = \theta_0$ $against$ H_1: $\theta \neq \theta_0$:~

Here $\Omega_0 = \{\theta_0\}$ and Ω= {0, θ> 0} $[\because e^{-\theta_0 n\bar{x}+\frac{1}{\bar{x}}n\bar{x}}$

$= e^{-\theta_0 n\bar{x}+n}$]

The likelihood ratio is

$$\lambda = \frac{\sup_{\theta \epsilon \Omega_0} L(\underset{\sim}{x};\theta)}{\sup_{\theta \epsilon \Omega} L(\underset{\sim}{x};\theta)} = \frac{L(\underset{\sim}{x};\theta_0)}{(\hat{\theta})^n . e^{-\theta \sum_{i=1}^{n} x_i}}$$

Where, $\hat{\theta} = \frac{1}{x}$ is the MLE of θ under Ω.

Here, $\lambda = (\theta_0\bar{x})^n . e^{-n(\theta_0\bar{x}-1)}$

$= y^n . e^{-n(y-1)}$, where $y = \theta_0\bar{x}$

Now, $\frac{d\lambda}{dy} = y^n . e^{-n(y-1)}$ (-n)$+ny^{n-1} . e^{-n(y-1)}$

$= ny^{n-1} . e^{-n(y-1)} \{1-y\}$

$$= \begin{cases} > 0 \ if \ y < 1 \\ < 0 \ if \ y > 1 \end{cases}$$

From graph, λ< c

$\Rightarrow$ y < k_1 or y > k_2

$\Rightarrow 2\ \theta_0 \sum_{i=1}^{n} x_i < a \ or \ 2\ \theta \sum_{i=1}^{n} x_i > b$

Where, $2nk_1 = a, 2nk_2 = b$

Here, the size αLRT is given by:

Reject H_0 iff λ > c iff $2\ \theta_0 \sum_{i=1}^{n} x_i \notin [a,b]$

Where ‘a’, ‘b’ are such that

$\alpha = PH_0\ [2\ \theta_0 \sum_{i=1}^{n} x_i \notin [a,b]]$

$= 1- PH_0[a \leq 2\ \theta_0 \sum x_i \leq b]$

$= 1- P\ [a \leq \chi^2_{2n} \leq b]$

$= 1- F\chi^2_{2n}(b) + \chi^2_{2n}(a)$ and $(k_1) = \lambda(k_2) \Rightarrow {k_1}^n . e^{-n(k_1-1)}$

$= {k_2}^n . e^{-n(k_2-1)}$

(II) To test $H_0 :\geq \theta_0$ *against* $H_1 : \theta > \theta_0$:-

Here, $\Omega_0 = \{\theta_0\}$ and $\Omega = \{ \geq \theta_0\}$

The likelihood ratio is

$$\lambda = \frac{\sup_{\theta \in \Omega_0} L(\underset{\sim}{x};\theta)}{\sup_{\theta \in \Omega} L(\underset{\sim}{x};\theta)} = \frac{{\theta_0}^n . e^{-\theta_0 \sum_{i=1}^{n} x_i}}{\sup_{\theta \geq \theta_0} \{L(\underset{\sim}{x};\theta)\}}$$

For $\theta > 0$, $L(\underset{\sim}{x};\theta)$ is maximum at $\theta = \frac{1}{\bar{x}} = \hat{\theta}$

$\therefore$ Sup $L(\underset{\sim}{x};\theta)$ $\theta \geq \theta_0$

$$= \begin{cases} (\hat{\theta})^n e^{-\hat{\theta} 2 x_i} , if\, \theta_0 < \hat{\theta} \\ {\theta_0}^n . e^{-\theta_0 \sum x_i} , if\, \theta_0 > \hat{\theta} \end{cases}$$

Now, $\lambda = \begin{cases} (\theta_0 \bar{x})^n . e^{-n(\theta_0 \bar{x} - 1)}, if\ \theta_0 \bar{x} < 1 \\ 1 \quad , if\ \theta_0 \bar{x} \geq 1 \end{cases}$

From graph, $\lambda <$ c(<1)

$\Rightarrow$ y< k

$\Rightarrow 2\theta_0 \sum_{i=1}^{n} x_i <$ a , say

The size α LRT is given

by: reject H_0 iff $\lambda <$ c, iff $2\ \theta_0 \sum_{i=1}^{n} x_i < a$

Where 'a' is such that $\alpha = PH_0\ [2\ \theta_0 \sum_{i=1}^{n} x_i < a]$

$\therefore \alpha = P\ [{\chi^2}_{2n} < a]$

$\Rightarrow a = {\chi^2}_{1-\alpha;2n}$

Therefore the size α LRT is given by:

Reject H_0 iff $\sum_{i=1}^{n} x_i < \frac{{\chi^2}_{1-\alpha;2n}}{2\theta_0}$.

(III) Hint:- $\lambda = \frac{\sup_{\theta \geq \theta_0} L(\underset{\sim}{x};\theta)}{\sup_{\theta \in R} L(\underset{\sim}{x};\theta)}$

The size α LRT is given by: Reject H_0 iff $\sum_{i=1}^{n} x_i > \frac{{\chi^2}_{\alpha;2n}}{2\theta_0}$.

Q75. let $X_1, X_2, \dots, X_n$ be a R.S. from N (θ, σ^2), σ known. Derive size α LRT for testing

(i) H_0:θ= θ_0 $against$ H_1: $\theta \neq \theta_0$

(ii) H_0:θ= θ_0 $against$ H_1: $\theta > \theta_0$

Show that the LRT's obtained are unbiased.

Solution:- the likelihood function is

$$L\ (\underset{\sim}{x};\ \theta) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n . e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2};\ where\ \theta\ \epsilon \mathbb{R}$$

(i) To test H_0:θ= θ_0 $against$ H_1: $\theta \neq \theta_0$:-

Here $\Omega_0 = \{\theta_0\}$ and Ω=$\{\theta_0: \theta \epsilon \mathbb{R}\}$

The likelihood ratio is $\lambda = \frac{\theta^{sup}\ \epsilon \Omega_0 L\ (\underset{\sim}{x};\theta)}{\theta^{sup\epsilon}\ \Omega L\ (\underset{\sim}{x};\theta)}$

$= \frac{L\ (\underset{\sim}{x}, \theta_0)}{\theta^{sup\epsilon}\ \mathbb{R}L\ (\underset{\sim}{x};\theta)}$

$= \frac{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n . e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta_0)^2}}{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n . e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2}}$

$= e^{-\frac{1}{2\sigma^2}\{\sum(x_i-\theta_0)^2 - \sum(x_i-\bar{x})^2\}}$

$= e^{-\frac{1}{2\sigma^2}.n\ (\bar{x}-\theta_0)^2}$

Note that λ< c

$\Rightarrow e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta_0)^2 < c}$

$\Rightarrow \frac{n(\bar{x}-\theta_0)^2}{\sigma^2} > c_1$

$\Rightarrow \left|\frac{\sqrt{n}\ (\bar{x}-\theta_0)}{\sigma}\right| >$ k, say.

The size α LRT is given by:

Reject H_0 iff λ> c iff $\left|\frac{\sqrt{n}\ (\bar{x}-\theta_0)}{\sigma}\right| >$ k, where k is such that

$\alpha = P_{H_0}\left[\left|\frac{\sqrt{n}\ (\bar{x}-\theta_0)}{\sigma}\right| > k\right] = P\ [|z| > k]$, z ~N (0, 1)

$\Rightarrow$ k = $\alpha/2$

(II) to test H_0:θ= θ_0 $against$ H_1: θ> θ_0:-

Here $\Omega_0 = \{\theta_0\}$ and Ω= $\{\theta_0: \theta \geq \theta_0\}$

The likelihood ratio is λ= $\frac{\theta^{sup} \epsilon \Omega_0 \mathrm{L}\ (\underset{\sim}{x};\theta)}{\theta^{sup\epsilon}\ \Omega \mathrm{L}\ (\underset{\sim}{x};\theta)}$

$= \frac{\mathrm{L}\ (\underset{\sim}{x};\theta)}{\theta^{\sup}\ {}^{\geq}\theta_0 \mathrm{L}\ (\underset{\sim}{x};\theta)}$

Here L $(\underset{\sim}{x};\ \theta)$ is maximum at θ= x̄= θˆ

Now, $\theta^{\sup}\ {}^{\geq}\theta_0 \mathrm{L}\ (\underset{\sim}{x};\ \theta) = \begin{cases} \mathrm{L}\ (\underset{\sim}{x};\ \hat{\theta}), \text{if } \bar{x} > \theta_0 \\ \mathrm{L}\ (\underset{\sim}{x};\ \theta_0)\ , \text{if } \theta_0 \geq \bar{x} \end{cases}$

Here, λ= $\begin{cases} e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta_0)^2}, if\, \theta_0 < \bar{x} \\ \quad 1 \quad , if\ \theta_0 \geq \bar{x} \end{cases}$

Note that, λ <c (< 1)

$\Rightarrow\ e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta_0)^2 < c}$, where $\theta_0 < \bar{x}$

$\Rightarrow \frac{n\ (\bar{x}-\theta_0)}{\sigma} > c_1$, where x̄>$\theta_0$

$\Rightarrow \frac{\sqrt{n}\ (\bar{x}-\theta_0)}{\sigma}$ > k , as $(\bar{x} - \theta_0)$> 0

The size α LRT is given by: Reject H_0 iff λ < c

Iff $\frac{\sqrt{n}\ (\bar{x}-\theta_0)}{\sigma}$ > k , where k is such that

α= $P_{H_0}\left[\frac{\sqrt{n}\ (\bar{x}-\theta_0)}{\sigma} > k\right]$

$= P\ [z > k]$, z ~N (0, 1)

The size LRT is given by: Reject H_0 iff x̄>$\theta_0 + \frac{\sigma}{\sqrt{n}}\ \alpha$,

Which is the UMP test for testing H_0: $\theta = \theta_0\ against\ H_1$: $\theta > \theta_0$ and is unbiased.

H_1: $\theta \neq \theta_0$ is $\beta(\theta)$= $P_{H_0}\left[\left|\frac{\sqrt{n}\ (\bar{x}-\theta_0)}{\sigma}\right| > \frac{\alpha}{2}\right]$

= 1- $\Phi\ (\frac{\sqrt{n}\ (\bar{x}-\theta_0)}{\sigma} + \frac{\alpha}{2})$+ $\Phi(\frac{\sqrt{n}\ (\theta_0-\theta)}{\sigma} - \frac{\alpha}{2})$

Note that, $\beta'(\theta)$= $\Phi(\frac{\sqrt{n}\ (\theta_0-\theta)}{\sigma} + \frac{\alpha}{2})(\frac{\sqrt{n}}{\sigma})$- $\Phi(\frac{\sqrt{n}\ (\theta_0-\theta)}{\sigma} - \frac{\alpha}{2})\times (\frac{\sqrt{n}}{\sigma})$; if $\theta > \theta_0$

Now, $\beta'(\theta)=\begin{cases} > 0 \; if \; \theta > \theta_0 \\ < 0 \; if \theta < \theta_0 \end{cases}$

Clearly, $\beta\ (\theta) > (\theta_0)\ \forall\ \theta \neq \theta_0$

$\Rightarrow$ power > size

i.e the LRT is unbiased

Q76. Let $X_1, \ldots, X_n$ be a R.S. from B (1, p) population. Derive a LRT of its size of $H_0: p = \{po\} against H_1: p \neq po$

Solution: - Here $\Omega_0 = \{p_0\}$ $and\ \Omega = \{p\ ;0 < p < 1\}$

The likelihood function is

$$L\ (\underset{\sim}{x}; p) = \begin{cases} p^{\sum x_i}(1-p)^{n-\sum x_i}, if\ x_i = 0, 1 \\ \quad 0 \quad , ow \end{cases}$$

The LR is

$\lambda = \frac{p^{sup}\ \epsilon\Omega_0 L\ (\underset{\sim}{x};p)}{p^{sup\epsilon}\ \Omega L\ (\underset{\sim}{x};p)} = \frac{L\ (\underset{\sim}{x};p_0)}{L\ (\underset{\sim}{x};\hat{p})}$, where

$\lambda = \frac{{p_0}^t(1-p_0)^{n-t}\hat{p}}{\hat{p}^t(1-\hat{p})^{n-t}}$ $, t = \sum x_i$

$= (\frac{np_0}{t})^t (\frac{n\ (1-p_0)}{n-t})^{n-t}$

Now, ln λ= tln($\frac{np_0}{t}$)+ (n-t)ln $\{\frac{n\ (1-p_0)}{n-t}\}$

And $\frac{d}{dt} ln\lambda$ = ln($\frac{np_0}{t}$)- ln $\{\frac{n\ (1-p_0)}{n-t}\}$

= | n($\frac{np_0}{t} . \frac{n-t}{n\ (1-p_0)t}$)

$= \begin{cases} > 0\ , if\ t < np_0 \\ < 0\ , if\ t > np_0 \end{cases}$

From graph, λ < c ⇒ t > $k_2\ or\ t < k_1$

The LRT of its size is given by;

Reject H_0 iff λ < c iff $\sum_{i=1}^{n} x_i < k_1$ or $\sum_{i=1}^{n} x_i > k_2 \ni \lambda\ (k_1) = \lambda\ (k_2)$

Q77. let X be a discrete random variable with P [X= -1]= P and P [x= k]= $(1-p)^2 P^k$, where P ϵ (0, 1)is unknown. Show that U(X)is an unbiased estimator of θ iff U (k)= ak , k = -1, 0, 1, 2, …. for some a.

Solution:- E [U(x)] = $\sum_{k=-1}^{\infty} a_k . P\ [X = k]$

= -ap + $\sum_{k=0}^{\infty} k(1-p)^2 P^k$

= -ap + a$(1-p)^2$[p+ $2p^2 + 3p^3 + \ ... \infty$]

=-ap +a$(1-p)^2$p$(1-p)^2$

= -ap+ ap

= 0.

Now, $\sum_{x=-1}^{\infty} u\ (x) f(x) = 0$

⇨ u (-1) P [X= -1]+ $\sum_{x=0}^{\infty} u\ (x) f(x) = 0$

⇨ u (-1) p + $\sum_{x=0}^{\infty} u\ (x)(1-p)^2 p^x$ = 0

⇨ p (u (-1)) + $(1-p)^2 \sum_{x=0}^{\infty} u\ (x)\ p^x$=0

⇨ $\sum_{x=0}^{\infty} u\ (x)\ p^x = \frac{-pu\,(-1)}{q^2}$

⇨ $\sum_{x=0}^{\infty} ux\ p^x$= -u (-1) p $(1-p)^2$

= -u (-1) p (1+ 2p +$3p^2 + \cdots$)

= -u (-1) [p+$2p^2 + 3p^3 + \cdots$]

= -u (-1) $\sum_{x=0}^{\infty} x\ p^x$ [comparing power series from both side]

⇨ u (x) = -u (-1)x

⇨ u(x) = ax.

Q78. f(x) = $\begin{cases} \boldsymbol{e^{-(x-\theta)}\ if\ x \geq \theta} \\ \boldsymbol{0 \quad ow} \end{cases}$

Find (a) MLE of θ (b) 95% C. I. for θ

Solution:- (a) let $X_1, \ldots, X_n$ be a R.S. from f(x). then,

$f_\theta(x) = \exp[-\sum_{i=1}^{n}(x_i - \theta)]. I\ X_{(1)} \geq \theta;$

= L(θ|$\underset{\sim}{x}$)

The likelihood function will be maximum when $\sum_i (x_i - \theta)$is minimum

i.e. , when θ is maximum.

i.e. $\hat{\theta}_{MLE} = X_{(1)}$

(b) CI will based on sufficient statistic

$$f_\theta(\underset{\sim}{x}) = \begin{cases} e^{-\sum_i x_i + n\theta} & ; X_{(1)} \geq \theta \\ 0 & ; ow \end{cases}$$

$$= e^{-\sum_i x_i + n\theta} \ \Phi\ (\theta, X_{(1)}) \text{ ; where } \Phi\ (\theta, X_{(1)}) = \begin{cases} 1 & if\ \theta \leq i \\ 0 & if\ ow \end{cases}$$

$\therefore f_\theta(\underset{\sim}{x}) = g_\theta\ (t).h\ (\underset{\sim}{x})$

Where, $g_\theta\ (t) = e^{n\theta}.\ \Phi\ (\theta, t)$ and $h\ (\underset{\sim}{x}) = e^{-\sum x_i}$

$\therefore$ by NF Factorization theorem $X_{(1)}$ is sufficient.

Now, the PDF of $X_{(1)}$ is given by $f_{X_{(1)}}(y) = ne^{-n\ (y-\theta)} I\ \ y \geq \theta$

Note that, it is a shifted exponential distribution,

$\therefore$ 2n $(X_{(1)} - \theta) \sim {\chi^2}_2$

$\therefore p_\theta\ \left[{\chi^2}_{2;1-\frac{\alpha}{2}} \leq 2n(X_{(1)} - \theta) \leq {\chi^2}_{2;\frac{\alpha}{2n}}\right] = 1 - \alpha$

$\Rightarrow P_\theta \left[\frac{{\chi^2}_{2;1-\frac{\alpha}{2}}}{2n} \leq X_{(1)} - \theta \leq X_{(1)}\ \right] = 1 - \alpha$

$\Rightarrow P_\theta\ \left[X_{(1)} - \frac{{\chi^2}_{2;1-\frac{\alpha}{2}}}{2n}\ \geq\ \theta\ \geq X_{(1)} - {\chi^2}_{2;\frac{\alpha}{2n}}\right] = 1 - \alpha$

$\therefore$ Confidence interval for θ is

$(X_{(1)} - {\chi^2}_{2;\frac{\alpha}{2n}}\ , X_{(1)} - \frac{{\chi^2}_{2;1-\frac{\alpha}{2}}}{2n})$

Q79. suppose $X_1, X_2, \ldots, X_n$ are i.i.d. N (θ, 1), $\theta_0 \leq\ \theta\ \leq\ \theta_1$, where $\theta_0 <\ \theta_1$ are two specified numbers. Find the MLE of θ and show that it is better than the sample mean $\bar{x}$ in the sence of having smaller mean squared error.

Solution:- L = $\prod_{i=1}^{n} f\ (x_i, \theta) = \frac{1}{(\sqrt{2\pi})^n}\ .e^{-\frac{1}{2}\sum(x_i - \theta)^2}$; $x_i \epsilon\ \mathbb{R}$

Here we wish to maximize L w.r.t. θ subject to the condition$\theta_0 \leq \theta \leq \theta_1$; L will be maximum iff $e^{\frac{1}{2}\sum(x_i-\theta)^2}$ is minimum. Iff $\sum(x_i - \theta)^2$ is minimum at $\theta\hat{}_{MLE} = \theta_0$

For x̄,

|nL = c-$\frac{1}{2}\sum(x_i - \theta)^2$

$$\frac{\partial|nL}{\partial\theta} = \sum(x_i - \theta) = 0$$

⇨ $\sum x_i = n\theta$

⇨$\theta\hat{}_{MLE} = \bar{x}$; where $\theta_0 \leq \bar{x} \leq \theta_1$ as $\theta_0 \leq \theta \leq \theta_1$

E $(X - \theta_0)^2 = E\ [X - \bar{x}) + (X - \theta_0)]^2$

= E $(X - \bar{x})^2$+ E $(X - \theta_0)^2$

⇨ E $(X - \theta_0)^2 \leq$ E $(X - \bar{x})^2$.

Q80. if T: $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation defined by

T (x, y, z) = (x+y+z, y+z, z) then find T^n(x, y, z).

Solution: - T^2(x, y, z) = T (x + y+ z, y+z, z)

= (x+ 2y + 3z, y+ 2z, z)

T^3(x, y, z) = T (x + 2y + 3z, y + 2z, z)

= (x+ 3y + 6z, y+ 3z, z)

⋮

T^n(x, y, z) = (x+ ny +$\frac{n\ (n+1)}{2}.z, y + nz, z$).

Q81. Let $X_1, X_2, \dots, X_n$ be i.i.d random variables with family f(θ ; x ; x $\epsilon\ \mathbb{R}$, $\theta\ \epsilon$ (0, 1) be the unknown parameter. Suppose that there exit an unbiased estimator T of θ based on sample size one, i.e.

E [T (X_1)] = θ. Assume that V (T (X_1)) < ∞.

(I) find the estimator V_n for θ, E (X_n) ∍ V_n is constant.

(II) Let s_n be the MVUE of θ based on $X_1, X_2, \dots, X_n$ s.t. $\text{lt}_{n\to\infty} V(s_n) \longrightarrow 0$

Solution:- (I) $X_i \sim f_\theta$ (x)

Y = T $(X_1) \sim f_\theta$(t)

Now, $\bar{Y} \longrightarrow$ E (Y_1)

i.e. $V_n = \frac{1}{n}\sum_{i=1}^{n}$ T $(X_i) \longrightarrow$ E [T$(X_1) = \theta$]

(II) E ($\bar{Y}$) = 0

$$V\ (s_n\ ) \leq V\ (V_n) - \frac{V\ (\mathrm{T}(X_1))}{n^2}$$

= $\frac{V\ (\mathrm{T}(X_1))}{n}$

⇨$\mathrm{lt}_{n\to\infty}\ V\ (s_n\ ) = 0$.

Q82. Let f (0)=0 then show that $\lim_{h\to 0}\frac{f\ (h)+f\ (-h)}{h^2} = f''\ (0)$

Solution:- f (h)= f(0)+hf'(0)+$\frac{h^2}{2!}f''\ (0)$+ $\frac{h^3}{3!}f'''\ (c)$ [∵ f(0)= 0]

Similarly, f(-h)= - hf'(0)+ $\frac{h^2}{2!}f''\ (0) - \frac{h^3}{3!}f'''\ (c)$

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f (h)+ f (-h)= $h^2 f''\ (0)$

∴ $f''\ (0) = \mathrm{lt}_{h\to 0}\frac{f\ (h)+f\ (-h)}{h^2}$.

Q83. Suppose x has a normal distribution with mean 0 and variance 25.

Let Y be an independent RV taking values -1 and +1 with equal probability ½ .define

S= X y + $\frac{X}{y}$, T = xy -$\frac{x}{y}$.

(I) Find the probability distribution of S.
(II) Find the probability distribution of ($\frac{S+T}{10})^2$

Solution:- (i) $F_S(s) = P\ [S\ \leq s]$

= P [S≤ $s \mid Y = -1]P\ [Y = -1] +\ P\ [S \leq s|Y = 1]\ P\ [Y = 1]$

= $\frac{1}{2}\ P\ [-2X\ \leq s] + \frac{1}{2}\ P\ [-2x \leq s]$

=P [x ≤ $\frac{s}{2}$], since 'X' is symmetrically distributed about '0',

= P [$\frac{x-0}{S}\ \leq\ \frac{s-0}{10}$]

$= \Phi(\frac{s}{10})$

$\therefore$ S ~ N (0, 10^2).

(Ii) s + T =2 xy

$(S+T)^2 = 4\,x^2y^2 = 4\,x^2$, since P $[Y^2 = 1]$=1

⇨ $(S+T)^2 = (\frac{X}{5})^2 \sim \ \chi^2$.

Q84. using an appropriate probability distribution or otherwise find the valu of $\lim_{n\to\infty} \frac{1}{2^{n/2}\,\Gamma(\frac{n}{2})} \cdot \int_{n+\sqrt{2}\,n}^{\infty} e^{-t/2}\ t^{n/2-1}\ dt$

Solution:- Let $\{x_n\}$ be a sequence of i.i.d. Random variables following Gamma $(\frac{1}{2}, \frac{1}{2})$

Here , f (x)= $\begin{cases} \frac{e^{-x/2}.x^{1/2-1}}{2^{1/2}\cdot\Gamma(\frac{1}{2})} & if\ x > 0 \\ 0 & ow \end{cases}$

Then $\sum_{k=1}^{n} X_k = S_n \sim Gamma(\frac{1}{2}, \frac{n}{2})$

E (S_n)= n, V(S_n)= 2n. by lindeberg- lavy central Limit theorem

$$\lim_{n\to\infty} P\left[\frac{S_n - E\,(S_n)}{\sqrt{V(S_n)}} \le x\right] = \ \Phi\ (x)$$

⇨ $\lim_{n\to\infty} P\left[\frac{S_n - n}{\sqrt{2n}} \le 1\right] = \ \Phi(1)$

⇨ $\lim_{n\to\infty} P\left[S_n\ \le n + \sqrt{2n}\,\right] = \Phi(1)$

⇨ $\lim_{n\to\infty} P\left[S_n > n + \sqrt{2n}\,\right] = 1 - \Phi(1)$

⇨ $\lim_{n\to\infty} \left\{\frac{1}{2^{n/2}\,\Gamma(\frac{1}{2})} \int_{n+\sqrt{2}\,n}^{\infty} e^{-t/2}\ t^{n/2-1}\ dt\right\} = \Phi(-1)$

Q85. letY_1, Y_2, Y_3, Y_4 has four uncorrelated r.v.s with E (Y_i) = iθ,

V (Y_i) = $i^2\sigma^2$, i= $1(1)^4$, where θ, σ> 0 are unknown parameter

Find the values of c_1, c_2, c_3, c_4 for which $\sum_{i=1}^{4} c_i Y_i$ is unbiased for   and has least variance.

Solution:- E $(\sum_{i=1}^{4} c_i Y_i)$= $(\sum_{i=1}^{4} i_\theta\, c_i)\theta$, $\sum c_i = 1$

Again, V $(\sum_i c_i Y_i)= \sum c_i^2\, i^2\sigma^2 = \sigma^2 = \sum i^2 c_i^2$

Now, $\sum c_i Y_i$ is unbiased.

So, $1 = (\sum_i ic_i)^2 \le (\sum_i i^2 c_i^2)(\sum_i 1)$, by C-S inequality.

'=' holds when ic_i = k= $\frac{1}{4}$ ⇨ $c_i = \frac{1}{4i}$.

Alternative way:-

$$X_i \sim (\theta, \sigma^2)$$

$$Y_i \sim (i_\theta, i^2\sigma^2)$$

$$\frac{Y_i}{i} \sim (\theta, \sigma^2)$$

$$\bar{Y} \xrightarrow{BLUE} \theta$$

$\frac{1}{4}\sum \frac{Y_i}{i}$ is BWE for θ

$\sum c_i Y_i$ is BWE for θ, where $c_i = \frac{1}{4i}$

Q86**. Let $X_1, X_2, \dots, X_n$ be independently distributed random variables with densities**

f $(x_i;\ \theta)$= $\begin{cases} e^{i_\theta - x_i} & ,if\ x_i \ge i_\theta \\ 0 & ,ow \end{cases}$ **[Here $x_i^{'} sare\ not\ random\ samples$]**

Find a one –dimensional sufficient statistic for θ.

Solution:- the joint PDF of $X_1, X_2, \dots, X_n$ is

$$\prod_{i=1}^{n} f(x_i;\ \theta) = \begin{cases} e^{\theta\sum_{i=1}^n x_i - \sum_{i=1}^n x_i} & ; if x_i \ge i_\theta, \forall\ i = 1(1)n \\ 0 & , \quad ow \end{cases}$$

$$= \begin{cases} e^{\frac{n(n+1)\theta}{2} - \sum_{i=1}^n x_i} & ; if\ \frac{x_i}{i} \ge \theta \forall\ i = 1(1)n \\ 0 & ;\quad ow \end{cases}$$

$$= \begin{cases} e^{\frac{n(n+1)\theta}{2} - \sum_{i=1}^n x_i} & ; if\ \min_i\{\frac{x_i}{i}\} \ge \theta \\ 0 & ; ow \end{cases}$$

$= e^{\frac{n(n+1)\theta}{2} - \sum_{i=1}^n x_i} . I\ (\theta, \min\{\frac{x_i}{i}\})$; where I(a,b)= $\begin{cases} 1 & if\ a \le b \\ 0 & if\ a > b \end{cases}$

$= e^{\frac{n(n+1)\theta}{2}} . I\ (\theta, \min\{\frac{x_i}{i}\}).\ e^{-\sum_{i=1}^{n} x_i}$

= g (T ($\underset{\sim}{x}$) ;θ). h($\underset{\sim}{x}$), where h($\underset{\sim}{x}$)= $e^{-\sum x_i}$

so, T ($\underset{\sim}{x}$) = $\min_i\{\frac{x_i}{i}\}$ is sufficient for θ, by NFFT.

Q87. If f(x) = $\boldsymbol{\frac{\theta a^{\theta}}{x^{\theta+1}}}$ $\boldsymbol{I\ a} < x < \infty, \theta > 0, a > 0$;

(a) Find UMVUE of θ, when a is known,

(b) Find UMVUE of a, when θ is known

Solution: - (a) from OPEF, the complete sufficient statistic is $\sum_{i=1}^{n} |nX_i$ and hence$\sum_{i=1}^{n} |n\frac{X_i}{a}$ will also be complete sufficient statistic.

Now, $|n\frac{X_i}{a}$ ~ Exp (θ)

$\therefore 2\theta \sum_{i=1}^{n} |n\frac{X_i}{a} \sim {\chi_{2n}}^2$

⇨ E $[\frac{1}{2\theta \sum_{i=1}^{n} |n\frac{X_i}{a}}] = \frac{1}{(2n-2)}$ [If X $\sim {\chi_n}^2$ E $(\frac{1}{X}) = \frac{1}{n-2}$]

⇨ E $[\frac{n-1}{\sum_i |n\frac{X_i}{a}}] = \theta$

⇓

Function of complete sufficient statistic

$\therefore \frac{n-1}{\sum_i |n\frac{X_i}{a}}$ is the required UMVUE.

(b) $f_a(\underset{\sim}{x}) = \frac{\theta^n a^{n\theta}}{(\prod_{i=1}^{n} X_i)^{\theta+1}}$. $\Phi(a, x)$; where $\Phi(a, x) = \begin{cases} 1, if\ a < x \\ 0,\ ow \end{cases}$

$\therefore g_a(t) = a^{n\theta} . \Phi(a, t)$ With T = $X_{(1)}$

$\therefore X_{(1)}$ is the sufficient statistic for θ.

Now, $f_{X_{(1)}}(t) = P\ [X_{(1)} \leq\ t]$= 1-P $[X_{(1)} > t]$= 1- $(P[X > t])^n$

Where, P [X > t]= $\int_t^{\infty} \theta a^{\theta} x^{-\theta-1}\, dx = \frac{a^{\theta}}{t^{\theta}}$

$\therefore f_{X_{(1)}}(t) = -n\theta \left(\frac{a}{t}\right)^{n\theta -1} . (-\frac{a}{t^2})$

= n . $\frac{a^{n\theta}}{t^{n\theta +1}} \; I_a > 0$

Now, if (t) is any arbitrary function of t, then E [ψ(t)]=0

⇨ $\int_a^{\infty} n\theta \frac{a^{n\theta}}{t^{n\theta +1}} \psi(t)dt = 0$

⇨ $\int_a^{\infty} \frac{\psi(t)}{t^{n\theta +1}} dt = 0$

g (t), say

⇨ G (∞)-G (a)=0

⇨ g (∞)-0 - g (a)-1=0

⇨ g (a)=0 ⇨ ψ(a)=0

$\therefore X_{(1)}$ is complete sufficient for θ

Let $X_{(1)} = T$

Now, E (T)= $\int_a^{\infty} t. n\theta \frac{a^{n\theta}}{t^{n\theta +1}} dt$

=$\frac{n\theta - a}{n\theta +1}$

$\therefore$ E [t. $\frac{n\theta -1}{n\theta}$]= a

$\therefore$ UMVUE of a is $\frac{n\theta -1}{n\theta} = (1 - \frac{1}{n\theta})$.

Q88. let $X_1, X_2, \dots, X_n$~ Rec (0, θ) with an unknown θ (1, ∞)

Suppose we only observe $z_i = \begin{cases} X_i & if X_i \geq 1 \\ 1 & if\ X_i < 1 \end{cases}$

Derive UMVUE of θ.

Solution:- let T ($X_{(n)}$) be an unbiased estimator of 0.

$\therefore$ E [T ($X_{(n)}$)]=0

⇨ $\int_0^{\infty} t(x_{(n)}) . \frac{nx\,(n)^{n-1}}{\theta^n} \; dx_{(n)} = 0$

⇨ $\int_0^\theta t(x_{(n)}).\ x(n)^{n-1} dx_{(n)} = 0$⇨ G ($\theta$)- G(0)=0

⇨ g (θ)= 0 ⇨ u ($x_{(n)}$). $x(n)^{n-1}$=0 ⇨ u ($x_{(n)}$)=0

∴ $X_{(n)}$ is complete sufficient for 0.

To find UE of θ based on$X_{(n)}$, let us consider the function,

h (ˆ$X_{(n)}$)= $\begin{cases} a \ \ if\ X_{(n)} < 1 \\ bX_{(n)} \ \ if\ X_{(n)} \geq 1 \end{cases}$

∴ E [h ($X_{(n)}$)] = θ [∵ h ($X_{(n)}$) is UE of θ]

⇨ a. P [$X_{(n)} \leq 1$]+ b $x_{(n)}.\frac{nx(n)^{n-1}}{\theta^n} dx_{(n)}$=$\theta$

⇨ a $\int_0^1 \frac{nx(n)^{n-1}}{\theta^n}\ dx_{(n)} + b\ \int_1^\theta n.\frac{x_{(n)}}{\theta^n}\ dx_{(n)} = \theta$

⇨ a. $\frac{n}{\theta^n}.\frac{1}{n} + \frac{b.n}{(n+1)\theta^n} = \theta$

⇨ a+ $\frac{bn}{n+1}(\theta^{n+1} - 1) = \theta^{n+1}$

⇨ a+ $\frac{bn}{n+1}\theta^{n+1}$ -$\frac{bn}{n+1}$= θ^{n+1}

∴ $\frac{bn}{n+1} = 1, a = 1, \frac{bn}{n+1} = 1$. [Equating coefficients of θ]

∴ h (X) = $\begin{cases} 1 \ \ ifX_{(n)} < 1 \\ \frac{n+1}{n}X_{(n)} \geq 1 \end{cases}$ is UMVUE of θ.

Q89. Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution having pdf

f(x; x_0, α)= $\begin{cases} \frac{\alpha x_0}{x^{\alpha+1}}\ for\ x > \ x_0, \\ 0 \qquad\qquad ow \end{cases}$

Where, x_0, > 0, α> 0. Find the maximum likelihood estimator of α if x_0 is known.

Solution: - Likelihood function, L (x, α) is given by,

L (x, α) = $\frac{\alpha^n {x_0}^{\alpha n}}{{x_1}^{(\alpha+1)}{x_2}^{(\alpha+1)}\dots{x_n}^{(\alpha+1)}}$

⇨ $\log L = n\log\alpha + \alpha n\log x_0 - (\alpha+1)\sum x_i$

⇨ $\frac{1}{L}.\frac{\partial L}{\partial\alpha}$= $\frac{n}{\alpha} + \ n\log x_0 - \sum x_i \frac{\partial L}{\partial\alpha} = 0$

⇨ $\frac{n}{\alpha}$+n|n$x_0 = \sum x_i$

⇨ $\alpha^{-1} = \frac{\sum x_i}{n} - |\text{n}x_0$

⇨ α= $\frac{1}{\frac{\sum x_i}{n} - |\text{n}x_0}$

Thus MLE α is given by α= $\frac{1}{\frac{\sum x_i}{n} - |\text{n}x_0}$

Q90. A fair coin is flipped 2n times. Find the probability that it comes up heads more ofter than it comes up tail.

Solution: - P (No. of Heads > No. of Tails)+ P (No. of Heads= No. of Tails)+ P (No. of Heads < No. of Heads < No. of Tails)=1.

Assuming you are tossing a fair coin, by symmetry, we also have that

P (No. of Heads > No. of Tails)

= P (No. of Heads < No. of Tails)

If we want to get k heads in 2n tosses, where the probability of getting a head is P then the probability is.

$$\binom{2n}{k} P^k (1-P)^{2n-k}$$

In our case, if we want the number of heads to be the some as number of tails then k=n and if we are tossing a fair coin then P =$^1/_2$. Hence, we get

P (No. of Heads=No. of Tails)

= $\binom{2n}{n} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^n$

= $\frac{1}{2^{2n}} \binom{2n}{n}$

Hence, we get that P (No. of Heads > No. of Tails)

= P (No. of Heads < No. of Tails)

= $\frac{1 - \frac{\binom{2n}{n}}{2^{2n}}}{2} = \frac{1}{2} - \frac{\binom{2n}{n}}{2^{2n+1}}$.