Q1. If $a_1 < a_2 < \ldots < a_m$, $b_1 < b_2 < \ldots < b_n$ and also $\sum_{i=1}^{m} |a_i - x| = \sum_{j=1}^{n} |b_j - x|$, where $x$ is any real number then prove that $a_i = b_j$ for all $i$ and $n = m$.

Solution: Let $f(x) = |a_1 - x| + |a_2 - x| + \ldots + |a_m - x|$
And $g(x) = |b_1 - x| + |b_2 - x| + \ldots + |b_n - x|$.
Then we know only points of non-differentiability of $f(x)$ is $a_1, a_2, \ldots, a_m$, and only points of non-differentiability of $g(x)$ is $b_1, b_2, \ldots, b_n$.
Since, $m, n$ are finite numbers and also given that $f(x) = g(x)$
So, we may write, $h \Rightarrow f'(a_i) = g'(a_i)$
And also, $LHL \{f'(a_i)\} = LHL \{g'(a_i)\}$
But as $f(x)$ is non-differentiable at $x = a_i$,
So, $LHL \{f'(a_i)\} \neq RHL \{f'(a_i)\}$, $LHL \{g'(a_i)\} \neq RHL \{g'(a_i)\}$
So, $g(x)$ is also not differentiable at $x = a_i$.
Now, since both the functions are equal so the points of discontinuity are same so $m = n$.
To show the another part, we need to show $a_i = b_i$.
In a similar way, we can say, for any given $b_r$ there exists $a_p$
Such that $b_r = a_p$.
So, $\{a_1, a_2, \ldots, a_m\}$ and $\{b_1, b_2, \ldots, b_n\}$ has one-to-one and onto correspondence.
Therefore, $m = n$ and every $a_i = b_j$ if $i = j$.

Q2. Suppose $w_1$ and $w_2$ are subspaces of $\mathbb{R}^4$ spanned by \{(1, 2, 3, 4), (2, 1, 1, 2)\} and \{(1, 0, 1, 0), (3, 0, 1, 0)\} respectively. Find a basis of $w_1 \cap w_2$. Also find a basis of $w_1 + w_2$ containing \{(1, 0, 1, 0), (3, 0, 1, 0)\}. ($\mathbb{R}$ : The set of all real numbers)

Solution: $w_1 = \{(1, 2, 3, 4), (2, 1, 1, 2)\} \quad w_2 = \{(1, 0, 1, 0), (3, 0, 1, 0)\}$
Now we will calculate \( \text{dim}(w_1 \cup w_2) \) which is equal to number of independent rows in

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 1 & 1 & 2 \\
1 & 0 & 1 & 0
\end{pmatrix}
\]

i.e. \( \text{Rank}(A)=4 \).

Now, \( \text{dim}(w_1 \cup w_2)=\text{dim}w_1+\text{dim}w_2-\text{dim}(w_1 \cap w_2) \)

\[\Rightarrow 4 = 2 + 2 - \text{dim}(w_1 \cap w_2)\]

\[\Rightarrow \text{dim}(w_1 \cap w_2)=0.\]

i.e. basis of \( (w_1 \cap w_2)=\{(0, 0, 0, 0)\} \)

\[\Rightarrow \psi^4 = w_1 \oplus w_2 \]

\[\Rightarrow \text{basis of } w_2 \text{ can be extended to form basis of } w_1 + w_2 \text{ which is given by} \]

\[= \{(1, 0, 1, 0), (3, 0, 1, 0), (0, 1, 0, 0), (0, 0, 0, 1)\} \]

Q3. Two players \( p_1 \) and \( p_2 \) are playing the final of a chess championship, which consist of a series of matches. Probability of \( p_1 \) winning a match is \( \frac{2}{3} \) and for \( p_2 \) is \( \frac{1}{3} \). The winner will be one who is ahead by 2 games as compared to the other player and wins at least 6 games. Now, if the player \( p_2 \) wins first four matches, find the probability of \( p_1 \) winning the championship.

Solution:- \( p_1 \) can win in the following mutually exclusive ways:

(a) \( p_1 \) wins the next six matches.

(b) \( p_1 \) wins five out of next six matches, so that after next six matches score of \( p_1 \) and \( p_2 \) are tied up. This is continued up to ‘2n’ matches (n≥ 0) and finally \( p_1 \) wins 2 consecutive matches.

Now, probability of case (a) =\( \left(\frac{2}{3}\right)^6 \) and probability of tie after 6 matches (in case (b))=

\[=\left(\frac{2}{3}\right)^5 \cdot \frac{1}{3} = \frac{2^6}{3^6}.\]

Now probability that scores are still tied up after another next two matches=\[\frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9}. \]
[1st match is own by $p_1$ and 2nd by $p_2$, or, by reversively]

Similarly probability that scores are still tied up after another ‘2n’ matches $= (\frac{4}{9})^n$.

\[
\Rightarrow \text{Total probability of } p_1 \text{ winning the championship } \\
= \left(\frac{2}{3}\right)^6 + \frac{2^6}{3^5} \sum_{n=0}^{\infty} \left(\frac{4}{9}\right)^2 \left(\frac{2}{3}\right)^{2n} \\
= \left(\frac{2}{3}\right)^6 + \frac{2^6}{3^5} \left(\frac{2}{3}\right)^2 \left(\frac{1}{1-\frac{4}{9}}\right) \\
= \frac{17}{5} \left(\frac{2}{3}\right)^6 \\
= \frac{1088}{3645}.
\]

Q4. Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from a continuous distribution. The random variables are ranked in the increasing order of magnitude. $R_i$ be the rank of the $i$th sample. Find the correlation coefficient between $R_1$ and $R_2$.

Solution:- $R_i$ be the rank of $X_i$.

$R_i$ be the random variable such that $P(R_i = r_i) = \frac{1}{n}$; $ri=1(1)n$.

\[\therefore \sum_{i=1}^{n} R_i = \frac{n(n+1)}{2}, \text{ a constant quantity.}\n\]

And since $R_1, R_2, \ldots, R_n$ is identical random variable, now $R_i$ is th random variable and $\sum_{i=1}^{n} R_i$ is a constant.

\[\therefore \text{cov}(R_1, \sum_{i=1}^{n} R_i) = 0\]

\[\Rightarrow \text{cov}(R_1, R_1 + R_2 + \cdots + R_n) = 0\]

\[\Rightarrow \text{var}(R_1) + \text{cov}(R_1, R_2) + \cdots + \text{cov}(R_1, R_n) = 0\]

\[\Rightarrow \text{var}(R_1) + (n-1). \text{Cov}(R_1, R_2) = 0 \quad [\because R_i \text{’s are identically distributed}; \text{cov}(R_i, R_j) = \text{cov}(R_i)]\]

\[\Rightarrow \text{cov}(R_1, R_2) = \frac{\text{var}(R_1)}{(n-1)}\]

\[= \frac{n^2-1}{n^2} = \frac{(n+1)}{12}\]
\[ \therefore \rho = \frac{\text{cov}(R_1, R_2)}{\sqrt{\text{var}(R_1) \text{var}(R_2)}} = \frac{n+1}{12} \frac{(n+1)(n-1)}{12} = -\frac{1}{(n+1)}. \]

Q5. Let \( Y_1, Y_2, Y_3 \) be i.i.d. continuous r.v.s for \( i=1, 2 \). Define \( U_i \) as \( U_i = 1 \) if \( Y_{i+1} > Y_i \)
= 0 otherwise

Find the mean and variance of \( U_1 + U_2 \).

Solution:
\[
\begin{align*}
E(U_i) &= 1 \cdot P[Y_{i+1} > Y_i] = \frac{1}{2} \\
E(U_i)^2 &= 1^2 \cdot P[Y_{i+1} > Y_i] = \frac{1}{2} \\
V(U_i) &= \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \\
E(U_1 + U_2) &= \frac{1}{2} + \frac{1}{2} = 1.
\end{align*}
\]

\[
E(U_1 U_2) = 1 \cdot 1 \cdot P[Y_2 > Y_1, Y_3 > Y_2] = P[Y_3 > Y_2 > Y_1] = \frac{1}{6}.
\]

\[
\text{Cov}(U_1, U_2) = E(U_1 U_2) - E(U_1) E(U_2) = \frac{1}{12}.
\]

\[
\therefore V(U_1 + U_2) = V(U_1) + V(U_2) + 2 \text{cov}(U_1, U_2) = \frac{1}{3}.
\]

Q6. Let \( X \) and \( Y \) are i.i.d. with \( P[X= x] = \frac{1}{x} - \frac{1}{x+1}, x=1, 2, \ldots \)

Find \( E[\text{Min}(X, Y)] \).

Solution:
Let \( T= \text{min}(X, Y) \).

\[
P[T= t] = P[X= t, Y>t] + P[Y= t, X>t] + P[X= t, Y=t]
= P[X=t]P[Y > t] + P[X > t]P[Y=t] + P[X=t]P[Y= t]
\]

Now, \( P[Y \leq t] = P[Y=1] + P[Y=2] + \ldots + P[Y=t] \)
\[
= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{t} - \frac{1}{t+1})
= 1 - \frac{1}{t+1}
\]

\(
\therefore P[Y > t] = \frac{1}{t+1}
\)
Similarly, \( P[X > t] = \frac{1}{t+1} \)

Hence, \( P[T = t] = \left( \frac{1}{t} - \frac{1}{t+1} \right) \cdot \frac{1}{t+1} + \left( \frac{1}{t} - \frac{1}{t+1} \right) \cdot \frac{1}{t+1} + \left( \frac{1}{t} - \frac{1}{t+1} \right)^2 \)

\[ \Rightarrow \frac{1}{t(t+1)^2} + \frac{1}{t^2(t+1)} \]

\[ \therefore E(T) = \sum_{t=1}^{\infty} \frac{1}{(t+1)^2} + \sum_{t=1}^{\infty} \frac{1}{t(t+1)} \]

\[ = \left( \frac{\pi^2}{6} - 1 \right) + \sum_{t=1}^{\infty} \left( \frac{1}{t} - \frac{1}{t+1} \right) \]

\[ = \frac{\pi^2}{6} - 1 + 1 \]

\[ = \frac{\pi^2}{6} \]

Q7. Let \( P[X_n = -n^p] = \frac{1}{2} = P[X_n = n^p] \)

Show that WLLN holds for the sequence \( \{X_n\} \) of independent R.V.’s if \( P < \frac{1}{2} \)

Solution :- here, \( \mu_k = E(X_k) = 0 \)

\( \sigma_k^2 = V(X_k) = E(X_k)^2 = (-k^p)^2 \cdot \frac{1}{2} + (k^p)^2 \cdot \frac{1}{2} \)

\[ = k^{2p}, \quad k \in \mathbb{N} \]

Now, \( \frac{1}{n^2} \sum_{k=1}^{n} \sigma_k^2 = \frac{1}{n^2} \sum_{k=1}^{n} k^{2p} < \frac{1}{n^2} \int_{1}^{n} x^{2p} \, dx \)

\[ = \frac{n^{2p+1} - 1}{n^2(2p+1)} \]

Now, \( 0 \leq \frac{1}{n^2} \sum_{k=1}^{n} \sigma_k^2 \leq \frac{n^{2p+1} - 1}{n^2(2p+1)} < \frac{n^{2p-1}}{(2p+1)} \rightarrow 0 \) as \( n \rightarrow \infty \) \quad \text{[if} \ 2P-1 < 0, \ \text{if} \ P < \frac{1}{2} \]

\[ \Rightarrow \text{if} \ P < \frac{1}{2}, \ \frac{1}{n^2} \sum_{k=1}^{n} \sigma_k^2 \rightarrow 0 \) as \( n \rightarrow \infty \)

Hence, \( \{X_n\} \) obeys WLLN if \( P < \frac{1}{2} \).
Q8. If $X, Y \sim N(0, 1)$, Find the distn of $U = \frac{XY}{\sqrt{X^2 + Y^2}}$ and $V = \frac{X^2 - Y^2}{\sqrt{X^2 + Y^2}}$.

Solution:

$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}, (x, y) \in \mathbb{R}^2$

Let, $x = r\cos\theta$, $y = r\sin\theta$,

Here, $0 < r < \infty$, $0 < \theta < 2\pi$,

$\therefore J = r$,

The PDF of $(r, \theta)$ is ______

$g(r, \theta) = \begin{cases} r e^{-\frac{r^2}{2} - \frac{1}{2\pi}} & 0 < r < \infty \text{ and } 0 < \theta < 2\pi \\ 0 & \text{otherwise} \end{cases}$

Here, $u = r\sin\theta \cos\theta = \frac{r}{2} \sin 2\theta$

And $v = r\cos 2\theta$

Clearly, $(U, V) \in \mathbb{R}^2$

$J_1 = \frac{\partial(r,\theta)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(r,\theta)}} = \frac{1}{\frac{\partial}{\partial (r,\theta)} \begin{vmatrix} \frac{\partial}{\partial \theta} \sin \theta & \frac{\partial}{\partial \theta} \cos \theta \\ \frac{\partial}{\partial \theta} \cos 2\theta & -2\sin \theta \end{vmatrix}} = \frac{1}{r} = J_2$

Here, $(2u)^2 + v^2 = r^2$ [ a pair $(u,v)$ is obtained, for two pairs: $(r, \theta), (r, \theta + 2\pi)$. The transformation is not one-to-one]

$\Rightarrow r = \sqrt{4u^2 + v^2}$

The PDF of $(U, V)$ is

$f_{U,V}(u,v) = \frac{2}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{u^2}{2}}}{\sqrt{4u^2 + v^2}} \cdot (\sqrt{4u^2 + v^2}) \cdot \frac{1}{\sqrt{4u^2 + v^2}}, (u, v) \in \mathbb{R}^2$

$= \frac{1}{2\sqrt{2\pi}} \cdot e^{-\frac{u^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{v^2}{2}} ; (u, v) \in \mathbb{R}^2$

$= f_U(u) \cdot f_V(v), u, v \in \mathbb{R}$

Hence, $U \sim N(0, \frac{1}{4})$ and $V \sim N(0, 1)$, independently.
Q9. Let $X_1, ..., X_n$ be a R.S. from $f(x; \mu, \sigma) = \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}} ; x \in \mathbb{R}$, where $\mu \in \mathbb{R}, \sigma > 0$. Find the MLE of $\mu$ and $\sigma$.

Solution:– The log-likelihood function is

$$L(\mu, \sigma^2/x) = -n|n\sigma - \frac{1}{n} \sum x_i - \mu| ; \mu \in \mathbb{R}, \sigma > 0$$

[As $\sum x_i - \mu$ is not differentiable w.r.t. $\mu$, hence the derivative technique is not applicable for maximizing $nL$ w.r.t. $\mu$]

We adopt two stage maximization:

First fix $\sigma$, and then maximize $|nL$ for variation in $\mu$.

For fixed $\sigma$, $|nL$ is maximum, 

Iff, $\sum |x_i - \mu|$ is minimum 

Iff, $\mu = \bar{x} =$ the sample median 

$= \mu$, say.

Now, we maximize $|nL(\mu, \sigma^2/x) = -n|n\sigma - \frac{1}{n} \sum x_i - \mu|$, w.r.t. $\sigma$

Note that $\frac{\delta}{\delta\sigma}|nL(\mu, \sigma^2/x)$

$= \frac{n}{\sigma} + \frac{1}{\sigma^2} \sum |x_i - \mu|

= \frac{n}{\sigma^2} \{\sigma - \frac{1}{n} \sum |x_i - \mu| \}

\[
\begin{cases}
> 0, \sigma < \frac{1}{n} \sum |x_i - \mu| \\
< 0, \sigma > \frac{1}{n} \sum |x_i - \mu|
\end{cases}
\]

By, 1st derivative test, $|nL(\mu, \sigma^2/x)$ is maximum at $\sigma = \frac{1}{n} \sum_{i=1}^n |x_i - \mu|$

Hence, the MLE of $\mu$ and $\sigma$ are $\mu = \bar{x}, \sigma = \frac{1}{n} \sum |x_i - \bar{x}|.$
Q10. Find an MP test of testing $H_0$ such that $H_0 : X \sim f_0(x)$ against $H_1 : X \sim f_1(x)$ of its size, where

\[ f_0(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, x \in R \]

\[ f_1(x) = \frac{1}{2} e^{-|x|}, x \in R \]

S.T. the power of the test is greater than its size.

Solution:- By N-P lemma, for a particular value of $k$, the test

\[ \Phi(x) = \begin{cases} 1, & \frac{f_1(x)}{f_0(x)} > k \\ 0, & \text{ow} \end{cases} \]

Is an MP test of $H_0$ against $H_1$ of its size.

Now,

\[ \frac{f_1(x)}{f_0(x)} > k \]

\[ \Rightarrow e^{\frac{1}{2}(x^2 - 2|x|)} > k \]

\[ \Rightarrow e^{\frac{1}{2}(|x| - 1)^2 - 1} > k \]

\[ \Rightarrow (|x| - 1)^2 > k_2^2, k_2 > 0 \]

\[ \Rightarrow |x| - 1 < -k_2 \text{ or } |x| - 1 > k_2 \]

\[ \Rightarrow |x| < C_1 \text{ or } |x| > C_2 \]

[Alternative: - note that $f_1(x)$ has more probability in its tails and near 0 than $f_0(x)$ has. If either a very large or very small value of $x$ is observed, we suspect that $H_1$ is true rather than $H_0$. For some $C_1$ and $C_2$, we shall reject $H_0$ iff $\frac{f_1(x)}{f_0(x)} > k$]

To $|x| < C_1$ or $|x| > C_2$.

Hence, for some $C_1$ and $C_2$, the test

\[ \Phi(x) = \begin{cases} 1, & |x| < C_1 \text{ or } |x| < C_2 \\ 0, & \text{ow} \end{cases} \]

Is an MP test of $H_0$ against $H_1$ of its size.

Note, that, $\beta_\Phi(f_1) = P[1 \times 1 < C_1 \text{ or } 1 \times 1 < C_2 ]$

\[ = \int_{-w}^{w} f_1(x) dx, w = \{x: |x| < C_1 \text{ or } |x| > C_2 \} \]
\[ \int_w f_0(x) \, dx \text{, as } f_1(x) > f_0(x) \forall x \in \omega \]

= \mathcal{P}_{f_0}[1 \times 1 < C_1 \text{ or } 1 \times 1 < C_2]

= \beta_\phi(f_0) \cdot \text{(Proved)}. 