ISI MSTAT SUBJECTIVE SAMPLE PAPER

Q1. If a_1<a_2<………< a_m, b_1< b_2< ……<, b_n and also $\sum_{i=1}^{m}|ai - x|$=$\sum_{j=1}^{n}|bj - x|$, where x is any real number then prove that ai = bj for all i and n = m.

Solution: let f(x)=|a_1-x|+|a_2-x|+…..+|a_m-x|

And g(x) = |b_1-x|+|b_2-x|+…..+|b_n-x|.

Then we know only points of non-differentiability of f(x) is a_1,a_2,………, a_m,and only points of non-differentiability of g(x) is b_1, b_2,………, b_n,

Since, m, n are finite numbers and also given that f(x)=g(x)

So, we may write, $\frac{f(ai+h)-f(ai)}{h} = \frac{g(ai+h)-g(ai)}{h}$ $\forall$ h.

So, RHL{f'(ai)}=RHL {g'(ai)}

And also, LHL{f'(ai)}=LHL {g'(ai)}

But as f(x) is non-differentiable at x=ai,

So, LHL{f'(ai)}≠RHL {f'(ai)}, LHL{g'(ai)}≠RHL {g'(ai)→g(x) is also not differentiable at x=ai.

Now, since both the functions are equal so the points of discontinuity are same so m=n.

To show the another part, we need to show ai=bi.

In a similar way, we can say, for any given $b_r$there exists a_p

Such that b_r=a_p.

So, {a_1,a_2,………, a_m} and {b_1, b_2,………, b_n} has one –to-one and onto correspondence.

Therefore, m=n and every a_i = b_j if i = j.

Q2. Suppose w_1 and w_2 are subspaces of Ψ^4 spanned by {(1, 2, 3, 4) , (2, 1, 1, 2)} and {(1, 0, 1, 0), (3, 0, 1, 0)} respectively. Find a basic of $w_1 \cap w_2$. Also find a basis of w_1 + w_2 containing {(1, 0, 1, 0), (3, 0, 1, 0)}. (Ψ : The set of all real numbers)

Solution: w_1 ={(1, 2, 3, 4) , (2, 1, 1, 2)} $\quad$ $w_2 = \{(1, 0, 1, 0), (3, 0, 1, 0)\}$

Now we will calculate dim($w_{1\upsilon}w_2$) which is equal to number of independent rows in

1	2	3	4
2	1	1	2
1	0	1	0

i.e. Rank(A)=4.

Now, dim($w_{1\cup}w_2$)=dimw_1+dimw_2-dim($w_1 \cap w_2$)

⇨ 4 =2 + 2 - dim ($w_1 \cap w_2$)

⇨ dim ($w_1 \cap w_2$)=0.

i.e. basis of ($w_1 \cap w_2$)={(0, 0, 0, 0)}

⇨Ψ^4 = $w_{1\oplus}w_2$

⇨ basis of w_2 can be extended to form basis of $w_1 + w_2$ which is given by

= {(1, 0, 1, 0), (3, 0, 1, 0), (0, 1, 0, 0), (0, 0, 0, 1)}

Q3. Two players p_1 and p_2 are playing the final of a chess championship, which consist of a series of matches. Probability of p_1 winning a match is $^2/_3$and for p_2 is $^1/_3$. The winner will be one who is ahead by 2 games as compared to the other player and wins at least 6 games. Now, if the player p_2 wins first four matches, find the probability of p_1 winning the championship.

Solution:- p_1 can win in the following mutually exclusive ways:

(a) p_1 wins the next six matches.
(b) p_1 wins five out of next six matches, so that after next six matches score of p_1 and p_2 are tied up. This is continued up to '2_n' matches (n≥ 0) and finally p_1 wins 2 consecutive matches.

Now, probability of case (a) =$(\frac{2}{3})^6$ and probability of tie after 6 matches (in case (b))=

=$(\frac{2}{3})^5(\frac{1}{3})$ =$\frac{2^6}{3^5}$.

Now probability that scores are still tied up after another next two matches=$\frac{2}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{2}{3}=\frac{4}{9}$.

[1st match is own by p_1 and 2nd by $p_{2,}$ or , by reversively]

Similarly probability that scores are still tied up after another '2n' matches=$(\frac{4}{9})^n$.

⇨Total probability of p_1 winning the championship

$= (\frac{2}{3})^6 + \frac{2^6}{3^5} (\sum_{n=0}^{\infty} (\frac{4}{9})^2 \ (\frac{2}{3})^2)$

$= (\frac{2}{3})^6 + \frac{2^6}{3^5} (\frac{2}{3})^2 (\frac{1}{1-\frac{4}{9}})$

$= \frac{17}{5} (\frac{2}{3})^6$

$= \frac{1088}{3645}$.

Q4. Let $X_1, X_2, \dots, X_n$ be a random sample drawn from a continuous distribution. The random variables are ranked in the increasing order of magnitude. R_i be the rank of the ith sample. Find the correlation coefficient between R_1 and R_2.

Solution:- R_i be the rank of X_i.

R_i be the random variable such that P($R_i = r_i$) =$\frac{1}{n}$; ri=1(1)n .

$\therefore \sum_{i=1}^{n} R_i = \frac{n(n+1)}{2}$, a constant quantity.

And since $R_1, R_2, \dots, R_n$ is identical random variable, now R_i is th random variable and $\sum_{i=1}^{n} R_i$ is a constant.

$\therefore$ cov($R_1, \sum_{i=1}^{n} R_i$)=0

⇨ cov($R_1, R_1 + R_2 + \cdots + R_n$,)=0

⇨var(R_1) +cov(R_1, R_2) + ….+cov(R_1, R_n)=0

⇨var(R_1)+(n-1). Cov(R_1, R_2)=0 [∵R_i's are identically distributed; cov(R_i, R_j)= cov(R_i)]

⇨cov(R_1, R_2)=-$\frac{var(R_i)}{(n-1)}$

$= -\frac{\frac{n^2-1}{12}}{(n-1)} = -\frac{(n+1)}{12}$

$\therefore$ **ρ** = $\frac{cov(R_1,R_2)}{\sqrt{var(R_1)var(R_2)}} = \frac{-\frac{n+1}{12}}{\frac{(n+1)(n-1)}{12}} = -\frac{1}{(n+1)}$.

Q5. Let Y, Y_2, Y_3 be i.i.d. continuous r.v.s for i=1, 2. Define U_i as U_i =1 if $Y_{i+1} > Y_i$

=0 ow

Find the mean and variance of $U_1 + U_2$.

<u>Solution</u>:- E(U_i)=1.P[$Y_{i+1} > Y_i$]=$\frac{1}{2}$

E $(U_i)^2$=1^2.P[$Y_{i+1} > Y_i$]=$\frac{1}{2}$

V (U_i) =$\frac{1}{2} - \frac{1}{4}$ =$\frac{1}{4}$

E ($U_1 + U_2$)= $\frac{1}{2} + \frac{1}{2}$ =1.

E (U_1U_2) =1.1.P[$Y_2 > Y_1, Y_3 > Y_2$] = P[Y_3>$Y_2 > Y_1$]=$\frac{1}{6}$.

Cov(U_1, U_2)= E(U_1U_2)-E(U_1)E(U_2) = -$\frac{1}{12}$

$\therefore$ V($U_1 + U_2$)=V(U_1)+V(U_2)+2cov(U_1, U_2) =$\frac{1}{3}$.

Q6. Let X and Y are i.i.d. with P[X= x] = $\frac{1}{x} - \frac{1}{x+1}$, x=1, 2, …

Find E[Min(X, Y)].

<u>Solution</u>:- Let T= min (x, Y).

P[T= t]= P[X=t, Y>t]+ P[Y=t, X>t] + P[x=t, Y=t]

=P[x=t]P[Y> t]+ P[X> t]P[Y=t]+p [X=t]P[Y= t]

Now, P[Y≤t]= P[Y=1]+ P[Y=2]+…+P[Y=t]

=(1-$\frac{1}{2}$) + ($\frac{1}{2} - \frac{1}{3}$) + …+ ($\frac{1}{t} - \frac{1}{t+1}$)

= 1- $\frac{1}{t+1}$

⇨ P[Y>t]= $\frac{1}{t+1}$

Similarly, P[X> t]= $\frac{1}{t+1}$

Hence, P[T=t]=($\frac{1}{t} - \frac{1}{t+1}$). $\frac{1}{t+1}$ + $\frac{1}{t} - \frac{1}{t+1}$). $\frac{1}{t+1}$ + ($\frac{1}{t} - \frac{1}{t+1})^2$

=$\frac{1}{t(t+1)^2} + \frac{1}{t^2(t+1)}$

∴ E(T) =$\sum_{t=1}^{\infty} \frac{1}{(t+1)^2} + \sum_{t=1}^{\infty} \frac{1}{t\,(t+1)}$

=($\frac{\pi^2}{6} - 1$) + $\sum_{t=1}^{\infty}$ ($\frac{1}{t} - \frac{1}{t+1}$)

=$\frac{\pi^2}{6}$ -1 + 1

=$\frac{\pi^2}{6}$

Q7. Let P [$X_n = -n^P$]= $\frac{1}{2} = P[X_n = n^P]$

Show that WLLN holds for the sequence {X_n} of independent R.V.'s if P < $\frac{1}{2}$

Solution :- here, $\mu_k = E\ (X_k) = 0$

σk^2= V (X_k) $= E(X_k)^2 = (-k^P)^2.\frac{1}{2} + (k^P)^2.\frac{1}{2}$

= k^{2P}, k ∈ℕ

Now, $\frac{1}{n^2} \sum_{k=1}^{n} \sigma k^2 = \frac{1}{n^2} \sum_{k=1}^{n} k^{2P} < \frac{1}{n^2} \int_1^n x^{2P} dx$

= $\frac{n^{2p+1}-1}{n^2(2P+1)}$

Now, 0 ≤ $\frac{1}{n^2} \sum_{k=1}^{n} \sigma k^2 < \frac{n^{2p+1}-1}{n^2(2P+1)} < \frac{n^{2p-1}}{(2P+1)}$ ⟶0 as n ⟶∞ [if 2P-1< 0, if p < $\frac{1}{2}$]

⇒ if p < $\frac{1}{2}$, $\frac{1}{n^2} \sum_{k=1}^{n} \sigma k^2$ ⟶0 as n ⟶∞

Hence, {X_n} obeys WLLN if p < $\frac{1}{2}$.

Q8. If X, Y ~ N(0, 1), Find the distn of U= $\frac{XY}{\sqrt{X^2+Y^2}}$, and V = $\frac{X^2-Y^2}{\sqrt{X^2+Y^2}}$.

Solution:- $f_{x,Y}(x, y) = \frac{1}{2\pi} e^{-\frac{X^2+Y^2}{2}}$, (x, y)∈ $\mathbb{R}^2$

Let, x= rcosθ, y = rsinθ,

Here, 0< r<∞, 0< θ< 2π,

∴ J = r,

The PDF of (r, θ)is _____

g (r, θ)= $\begin{cases} re^{-\frac{r^2}{2}}.\frac{1}{2\pi} \text{ ,}0<r<\infty \text{ and } 0<\theta<2\pi \\ 0 \quad , \quad ow \end{cases}$

Here, u= rsinθcosθ = $\frac{r}{2} sin2\theta$

And v= rcos2θ

Clearly, (U, V) ∈ $\mathbb{R}^2$

$J_1 = \frac{\partial(\text{r},\theta)}{\partial(\text{u},\text{v})} = \frac{1}{\frac{\partial(\text{u},\text{v})}{\partial(\text{r},\theta)}} = \frac{1}{\begin{vmatrix} \frac{1}{2}sin^2\theta & \text{rcos}2\theta \\ \cos 2\theta & -2sin\theta \end{vmatrix}} = \frac{1}{r} = J_2$

Here, $(2u)^2 + v^2 = r^2$ [a pair(u,v)is a obtained, for two pairs: (r, θ), (r, θ+2π). The transformation is not one-to-one]

⇒ r = $\sqrt{4u^2 + v^2}$

The PDF of (U, V)is

$f_{UV}(u, v) = \frac{2 .e^{-\frac{4u^2+v^2}{2}}}{2\pi}.(\sqrt{4u^2+v^2})|\frac{1}{\sqrt{4u^2+v^2}}|$, (u, v) ∈ $\mathbb{R}^2$

$= \frac{1}{\frac{1}{2}\sqrt{2\pi}} . e^{-\frac{u^2}{2.\frac{1}{4}}} . \frac{1}{\sqrt{2\pi}} . e^{-\frac{v^2}{2}}$; (u, v) ∈ $\mathbb{R}^2$

$= f_U(u). f_V(u)$, u, v ∈ $\mathbb{R}$

Hence, U ~ N (0, $\frac{1}{4}$) and V ~ N (0, 1), independently.

Q9. Let $X_1, \dots, X_n$ be a R.S. from f (x; μ,)= $\frac{1}{2\sigma}\, e^{-\frac{|x-\mu|}{\sigma}}$; $x \in R$,where $\mu \in$R, σ > 0. Find the MLE of μ and σ.

Solution:- The log- likelihood function is

L (μ, $\sigma^2/\underset{\sim}{x}$) = -n|n2- n|n$\sigma$- $\frac{1}{\sigma}\sum|x_i - \mu|$; $\mu \in R, \sigma > 0$

[As$\sum|x_i - \mu|$is not differentiable w.r.t. μ, hence the derivative technique is not applicable for maximizing |nL w.r.t.μ]

We adopt two stage maximization:-

First fix σ, and then maximize |nL for variation in μ.

For fixed σ, |nL is maximum,

Iff, $\sum|x_i - \mu|$is minimum

Iff, μ=x̄= the sample median

= μ, say.

Now, we maximize |nL (μ, $\sigma^2/\underset{\sim}{x}$)=-n|n2-n|n$\sigma$- $\frac{1}{\sigma}\sum|x_i - \mu|, w.r.t.\sigma$

Note that $\frac{\delta}{\delta\sigma}$|nL ($\mu$, $\sigma^2/\underset{\sim}{x}$)

= $-\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum|x_i - \mu|$

= $-\frac{n}{\sigma^2}\{\sigma - \frac{1}{n}\sum|x_i - \mu|\}$

$$\begin{cases} > 0, \sigma < \frac{1}{n}\sum|x_i - \mu| \\ < 0, \sigma > \frac{1}{n}\sum|x_i - \mu| \end{cases}$$

By, 1st derivative test, |nL (μ, $\sigma^2/\underset{\sim}{x}$) is maximum at σ=$\frac{1}{n}\sum_{i=1}^{n}|x_i - \mu|$

Hence, the MLE of μ and σ are μ= x̄ , σ=$\frac{1}{n}\sum|x_i - \bar{x}|$.

Q10. Find an MP test of testing H_0 such that $H_0 : X \sim f_0(x)$ against $H_1 : X \sim f_1(x)$ of its size, where

$$f_0(x)= \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, x \in R$$

$$f_1(x)= \frac{1}{2}\, e^{-|x|}, x \in R$$

S.T. the power of the test is greater than its size.

Solution:- By N-P lemma, for a particular value of k, the test

$$\Phi(\text{x})=\begin{cases} 1 & , \frac{f_1(x)}{f_0(x)} > k \\ 0 & , ow \end{cases}$$

Is an MP test of H_0 against H_1 of its size.

Now, $\frac{f_1(x)}{f_0(x)} > k$

$\Rightarrow e^{\frac{1}{2}\{x^2-2|x|\}} > k_1$

$\Rightarrow e^{\frac{1}{2}}\{(|x|-1)^2-1\} > k_1$

$\Rightarrow (|x|-1)^2 > {k_2}^2, k_2 > 0$

$\Rightarrow$ |x|-1 < -k_2 or |x|-1>k_2

$\Rightarrow$ |x|< C_1 or |x|> C_2

[Alternative: - note that $f_1(x)$ has more probability in its tails and near 0 than $f_0(x)$ has. If either a very large or very small value of x is observed, we suspect that H_1 is true rather thanH_0. For some C_1 andC_2, we shall reject H_0 iff $\frac{f_1(x)}{f_0(x)} > k$

To |x|< C_1 or |x|<C_2.]

Hence, for some C_1 andC_2, the test

$$\Phi(\text{x}) = \begin{cases} 1, |\text{x}| < C_1 \text{ or } |\text{x}| < C_2 \\ 0, ow \end{cases}$$

Is an MP test of H_0 against H_1 of its size

Note, that, $\beta_{\Phi(}(f_1) = P[1 \times 1 < C_1 or\ 1 \times 1 < C_2\ ]$

$= \int_w^{f_1} f_1(x)dx\ , w = \{x: |x| < C_1\ or\ |x| > \ C_2\ \}$

$>\int_{w} f_0(x)dx$, as $f_1(x) > f_0(x) \forall x \epsilon w$

= $P_{f_0}[1 \times 1 < C_1 \; or \; 1 \times 1 < C_2]$

= $\beta_{\Phi}(f_0)$. (Proved).