MATHEMATICS PROBLEMS WITH SOLUTIONS

Algebra

Problem set

1. If $a_1 + a_2 + a_3 + \cdots \ldots \ldots a_n = 1, a_i > 0 \; for \; all \; a_i$ show that $\sum_{i=1}^{n} \frac{1}{a_i} \geq n^2$.
2. If $a_1, a_2, \ldots, a_n$ are all positive, then

$$\sqrt{a_1a_2} + \sqrt{a_1a_3} + \cdots + \sqrt{a_1a_n} + \sqrt{a_2a_3} + \sqrt{a_2a_4} \ldots + \cdots + \sqrt{a_2a_n} + \cdots + \sqrt{a_{n-1}a_1} + \sqrt{a_{n-1}a_2} + \cdots + \sqrt{a_{n-1}a_{n-1}} + \sqrt{a_{n-1}a_n} \leq \frac{n-1}{2}(a_1 + a_2 + \cdots + a_n).$$

3. If $w^3 + x^3 + y^3 + z^3 = 10$, show that 4

$$w^4 + x^4 + y^4 + z^4 \geq \sqrt[3]{2500}.$$

4. If x and y are real, solve the inequality

$$log_2 x + log_x 2 + 2\cos y \leq 0$$

5. Let P(x) $= x^2 + ax + b$ be a quadratic polynomial in which a and b are integers. Show that there is an integer M such that P(n). P(n+1) $= P(n)$ for any integer n.

6. Prove that the polynomial

$x^{9999} + x^{8888} + x^{7777} + \cdots + x^{1111} + 1$ is divisible by $x^9 + x^8 + x^7 + \cdots + x + 1$.

7. Find all integral solution of

$$x^3 + 5y^3 + 25z^3 - 15xyz = 0$$

8. Solve :

$$log_2 x + log_4 y + log_4 z = 2$$

$$log_3 y + log_9 z + log_9 x = 2$$

$$log_4 z + log_{16} x + log_{16} y = 2$$

9. Prove that the polynomial

$f(x) = x^4 + 26x^3 + 52x^2 + 78x + 1989$ cannot be expressed as a product of two polynomials p(x) and q(x) with integral coefficients of degree less than 4.

10. Find all positive integers x, y, z satisfying

$$x^{y^z} . y^{z^x} . z^{x^y} = 5xyz$$

11. Show that the set of polynomials

$$P = \{p_k | p_k(x) = x^{5k+4} + x^3 + x^2 + x + 1. k \epsilon N\}$$

Has a common non–trivial polynomial divisor.

12. If f is a polynomial with integer coefficients such that there exists four distinct integer $a_1, a_2, a_3, and \; a_4$ with $f(a_1) = f(a_2) = f(a_3) = f(a_4) = 1991$, then show that there exists no integer b, such that f(b) $=$1993.
13. Determine all the roots of the system of simultaneous equations x + y+ z $= 3, x^2 + y^2 + z^2 = 3 \; and \; x^3 + y^3 + z^3 = 3.$
14. Determine all pairs of positive integers (m, n) such that

$$\left(1 + x^n + x^{2n} + \cdots + x^{mn}\right) is \; divisible \; \left(1 + x + x^2 + \cdots + x^m\right).$$

15. Let x $= p, y = q, z = r \; and \; w = s$ be the unique solutions of the system of linear equations $x + a_i y + {a_i}^2 z + {a_i}^3 w = {a_i}^4, i = 1, 2, 3, 4.$ Express the solution of the following system in terms of p, q, r and s; $x + {a_i}^2 y + {a_i}^4 z + {a_i}^6 w = {a_i}^8, \quad i = 1, 2, 3, 4.$ (Assume the uniqueness of the solution)

16. **If P(x) is a polynomial of degree n such that $P(x) = 2^x$ for x =1, 2, 3, ….n + 1, find P(x+2).**
17. **What is the greatest integer, n for which there exists a simultaneous solution x to the inequalities k < $x^k < k+1, k = 1,2,3, \dots n$.**
18. **Let f be a function on the positive integers, i.e.,**

$f: N \to Z$ with the following properties:

$(a) f(2) = 2$

$(b) f(m \times n) = f(n)f(m)$ for all positive integers m and n.

$(c) f(m) > f(n)$ for $m > n$.

Find f(1998).

19. **A leaf is torn from a paper back novel. The sum of the remaining pages is 15,000. What are the page numbers of the torn leaf?**
20. **Show that a positive integer n good if there are n integers, positive or negative and not necessarily distinct, such that their sum and product both equal to n.**

Example 8 is as good as $= 4 \times 2 \times 1.1.1.1(-1).(-1) = 4+2+1+1+1+1+(-1)+(-1) = 8$

Show that the integers of the form (4k+1) where k ≥ 0 and $4l$ $(l \geq 2)$ are good.

21. **Show that for any triangle ABC, the following inequality is true**

$$a^2+b^2+c^2 > \sqrt{3}\max[|a^2-b^2|, |b^2-c^2|, |c^2-a^2|]$$

Where a, b, c are the sides of the triangle in the usual notation.

22. **Determine the largest number in the infinite sequence; 1; $\sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \dots, \sqrt[n]{n}$.**
23. **If $a_1 \geq a_2 \geq \cdots \geq a_n$ be real numbers such that**

$a_1{}^k + a_2{}^k + \cdots + a_n{}^k \geq 0$ for all integers k>0 and

$p = \max[|a_1|, |a_2|, \dots, |a_n|]$, prove that

$p = |a_1| = a_1$ and that $(x-a_1)(x-a_2)\dots(x-a_n) \leq x_n - a_1{}^n$.

24. **Let a > 2 be given and define recursively**

$$a_0 = 1, a_1 = a, a_{n+1} = \left(\frac{a_n{}^2}{a_{n-1}{}^2} - 2\right)a_n.$$

Show that for all k ϵ N, we have

$$\frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} < \frac{1}{2}\left(2 + a - \sqrt{a^2-4}\right)$$

25. **Let P(x) be a real polynomial function and**

$P(x) = ax^3 + bx^2 + cx + d$.

Prove if |P(x)| ≤ 1 for all x such that |x| ≤ 1 then |a|+|b|+|c|+|d| ≤ 7.

26. **Let a, b, c be real numbers with 0 < a, b, c < 1 and a+b+c = 2. Prove that**

$$\frac{a}{1-a}.\frac{b}{1-b}.\frac{c}{1-c} \geq 8$$

27. **If $a_0, a_1, \dots . a_{50}$ are the coefficients of the polynomial**

$$(1+x+x^2)^{25}$$

Prove that the sum $a_0, a_2 + \cdots a_{50}$ is even.

28. **Prove that the polynomial**

$$f(x) = x^4 + 26x^3 + 52x^2 + 78x + 1989$$

Cannot be expressed as product f(x) = $p(x)q(x)$ where p(x), q(x) are both polynomials with integral coefficients and with degree not more than 3.

29. Prove that

$$1 < \frac{1}{1001} + \frac{1}{1002} + \frac{1}{1003} + \cdots + \frac{1}{3001} < \frac{4}{3}$$

30. If x, y and z are three real numbers such that

$$x + y + z = 4 \text{ and } x^2 + y^2 + z^2 = 6$$

Then show that each of x, y, z lie in the closed interval [(2/3),2]. Can x attain the extreme value 2/3 and 2?

31. Let f(x) be a polynomial with integer coefficients. Suppose for five distinct integers a_1, a_2, a_3, a_4 and a_5 one has f(a_i) $= 2$ for $1 \le i \le 5$. Show that there is no integer b such that f(b) = 9.

32. Let f be a function defined on the set of non−negative integers and taking value in the same set. Suppose we are given that

(i) $x - f(x) = 19\left[\frac{x}{19}\right] - 90\left[\frac{f(x)}{90}\right]$ **for all non−negative integers x.**

(ii) $1900 < f(1900) < 2000$

Find all the possible values of f(1990). (Here [z] denotes the largest integer ≤ z; e.g., [3.145] = 3).

33. Solve for real x; $\frac{1}{[x]} + \frac{1}{[2x]} = \{x\} + \frac{1}{3}$, where [x] is the greatest integer less than or equal to x and {x} = $x - [x]$. [e.g., [3.4] = 3 and {3.4} = 0.4].

34. Define a sequence $\langle a_n \rangle_{n \ge 1}$ by

$a_1 = 1, a_2 = 2$ and $a_{n+2} = 2a_{n+1} - a_n + 2, \ge 1.$

Prove that for any m, $a_m a_{m+1}$ is also a term in the sequence.

35. Suppose a and b are two positive real numbers such that the roots of the cubic equation $x^3 - ax + b = 0$ are all real. If α is a root of this cubic with minimal absolute value prove that

$$\frac{b}{a} < \alpha \le \frac{3b}{2a}$$

36. Let a, b, c be three real numbers such that $1 \ge a \ge b \ge c \ge 0$. Prove that if l is a root of the cubic equation $x^3 + ax^2 + bx + c = 0$ (real or complex), then $|l| \le 1$.

Number Theory

37. Show the square of an integer cannot be in the form

4n+3 or 4n+2, where n ϵN.

38. Show that $n = 2^{m-1}(2^m - 1)$ is a perfect number, if $(2^m - 1)$ is a prime number.

39. When the numbers 19779 and 17997 are divided by a certain three digit number, they leave the same remainder. Find this largest such divisor and the remainder. How many such divisors are there?

40. Find the sum of all integers n, such that 1 $\le n \le 1998$ and that 60 divides $n^3 + 30n^2 + 100n$.

41. Prove by induction : $1^3 = 1, 2^3 = 3 + 5, 3^3 = 7 + 9 + 11, 4^3 = 13 + 15 + 17 + 19$ etc.

42. Prove by induction that if n ≥ 0, then $2^n > n^3$.

43. In a sequence 1, 4, 10, …….; $t_1 = 1, t_2 = 4, and\ t_n = 2t_{n-1} + 2t_{n-2}$ for n $\geq$ 3.

Show by second principle of mathematical induction that

$$t_n = \frac{1}{2}\left[\left(1+\sqrt{3}\right)^n + \left(1-\sqrt{3}\right)^n\right] for\ all\ n \in N.$$

44. Prove that for all natural numbers n, $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is divisible by 2^n.

45. A three digit number in base 11, when expressed in base 9, has its digits reversed. Find the number.

46. If n and k are positive integers and k > 1. Prove that

$$\left[\frac{n}{k}\right] + \left[\frac{n+1}{k}\right] \leq \left[\frac{2n}{k}\right]$$

47. How many zeroes does 6250! end with?

48. If n! has exactly 20 zeroes at the end, find n. How many such n are there?

49. Prove that [x] + [y] $\leq$ [x+y], x = $[x] + \{x\}$ and y = $[y] + \{y\}$, where both {x} and {y} are greater than or equal to 0.

50. Prove that [x] + [2x] + [4x] + [8x] + [16x] + [32x] = 12345 has no solution.

51. Find all the integral solutions of $y^2 = 1 + x + x^2$.

52. Find the product of

$$101 \times 10001 \times 100000001 \times \ldots \times (1000\ldots01)$$

Where the last factor has $2^7 - 1$ zeroes between the ones. Find the number of ones in the product.

53. Find the last two digits of $(56789)^{41}$.

54. Prove that $\frac{1.3.5.7\ldots99}{2.4.6.8\ldots100} < \frac{1}{10}$.

55. Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

56. Find all six digits numbers $a_1, a_2, a_3, a_4, a_5, a_6$ formed by using the digits 1, 2, 3, 4, 5, 6 once each such that the number $a_1, a_2, \ldots, a_k$ is divisible by k for $1 \leq k \leq 6$.

57. Find the number of all rational numbers $\frac{m}{n}$ such that

(i) 0 < $\frac{m}{n} < 1$,

(ii) m and n are relatively prime and

(iii) $m.n. = 25!$

58. Find the remainder when 4333^3 is divided by 9.

59. Let d be any positive integer not equal to 2, 5 or 13. Show that one can find distinct a, b in the set {2, 5, 13, d} such that $ab - 1$ is not a square.

60. Show that $1^{1997} + 2^{1997} + \cdots + 1996^{1997}$ is divisible by 1997.

61. Prove that $log_3 2$ is irrational.

62. Find all the ordered pairs of integers (x, z) such that $x^3 = z^3 + 721$.

63. Prove that for any natural number, n, E = $2903^n - 803^n - 464^n + 261^n$ is divisible by 1897.

64. Find all primes p for which the quotient $\left(2^{p-1} - 1\right) + p$ is a square.

65. $S = 1! + 2! + 3! + 4! + \cdots + 1997!$ Find the unit digit and tens digit of S.

66. All two digit numbers from 10 to 99 are written consecutively, that is $N =$ 101112…99. Show that $3^2|N$. From which other two digit number you should start so that N is divisible by (a) 3 (b) 3^2.

67. $N = 2^{n-1}(2^n - 1) and\ (2^n - 1)$ is a prime number. 1 < $d_1 < d_2 < \cdots < d_k = N$ are the divisors of N. Show that $\frac{1}{1} + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k} = 2$.

68. N $= P_1.P_2.P_3$ and $P_1, P_2\ and\ P_3$ are distinct prime numbers. If $\sum_{d/N} d = 3N$ (or $\mathbb{N}(N) = 3N$), show that $\sum_{i=1}^{N} \frac{1}{d_i} = 3$.

69. If n_1 and n_2 are two numbers, such that the sum of all the divisors of n_1 other than n_1 is equal to sum of all the divisors of n_2 other than n_2, then the pair (n_1, n_2) is called an amicable number pair.

Given : $a = 3.2^n - 1$,

$b = 3.2^{n-1} - 1$

And $c = 9.2^{2n-1} - 1$

Where a, b and c are all prime numbers, then show that ($2^n ab, 2^n c$) is an amicable pair.

70. Show that s(N) $= 4N\ when\ N =$30240.

71. Show that f($P_1^{a_1}.P_2^{a_1}) = f(P_1^{a_1}).f(P_2^{a_2}), where\ P_1\ and\ P_2$ are distinct prime.

72. Define $F(n) = \sum_{q|n} t_3(d) =$ cube of the number of divisors of d, i.e., F(n) is defined as the sum of the cubes of the number of divisors of the divisors of n.

73. Show that F $(P_1^{a_1} \times P_2^{a_2}) = F(P_1^{a_1}) \times F(P_2^{a_2})$.

74. Prove that F($P_1^{a_1}) = \{f(P_1^{a_1})\}^2$, where F and f are as defined in problems 56 and 62.

75. Sum of the cubes of the number of divisors of the divisors of a given number is equal to square of their sum. For example if N =18.

The divisors of 18 are 1, 2, 3, 6, 9, 18

No. of divisors of 18 are 1 2 2 4 3 6

Sum of the cubes of these divisors

$$1^3 + 2^3 + 2^3 + 4^3 + 3^3 + 6^3 = \left(1^3 + 2^3 + 3^3 + 4^3\right) + 2^3 + 6^3 = 100 + 224 = 324\ .$$

Square of the sum of these divisors $= (1 + 2 + 2 + 4 + 3 + 6)^2 = 18^2 = 324$.

76. Find all positive integers n for which $n^2 + 96$ is a perfect square.

77. There are n necklaces such that the first necklace contains 5 beads, the second contains 7 beads and, in general, the ith necklace contains i beads more than the number of beads in $(i - 1)$th necklace. Find the total number of beads in all the n necklaces.

78. Let a sequence $x_1, x_2, x_3, \ldots$, of complex numbers be defined by $x_1 = 0, x_{n+1} = x_n{}^2 - i$ for n > 1 where $i^2 = -1$. Find the distance of $x_{2000}\ from\ x_{1997}$ in the complex plane.

79. Find all n such than $n!$ has 1998 zeroes at the end of $n!$

80. Let f be a function from the set of positive integers to the set of real numbers {f : N → R} such that

(i) $f(1) = 1$

(ii) $f(1) + 2f(2) + 3f(3) + \cdots + nf(n) = n(n+1)f(n)$. Find f(1997).

81. Suppose f is a function on the positive integers, which takes integer values (i.e. $f: N \to Z$) with the following properties:

(a) $f(2) = 2$

(b) $f(m.n) = f(m).f(n)$

(c) $f(m) > f(n)$ if m > n.

Find f(1983).

82. Show that for

$f(m) = \frac{1}{8}[(3+2\sqrt{2})^{2m+1} + (3-2\sqrt{2})^{2m+1} - 6]$ both f(m)+1 and 2f(m)+1 are perfect squares for all m ϵ N by showing that f(m) is an integer.

83. Show that $n = \frac{1}{8} \times [(17+12\sqrt{2})^m + (17-12\sqrt{2})^m + 6]$ is an integer for all m ϵ N and hence, show that both (n−1) and (2n−1) are perfect squares for all m ϵ N.

84. A sequence of numbers a_n, n =1, 2, … is defined as follows:

$a_1 = \frac{1}{2}$ and for each $n \ge 2, a_n = \left(\frac{2n-3}{2n}\right) a_{n-1}$. Prove that $\sum_{k=1}^{n} a_k < 1$ $for\ all\ n \ge 1$.

85. Let T be the set of all triplets (a, b, c) of integers such that $1 \le a \le b \le c \le 6$. For each triplet (a, b, c) in T, take the number $a \times b \times c$ and add all these numbers corresponding to all the triplets in T. Prove that this sum is divisible by 7.

86. Find the least number whose last digit is 7 and which becomes 5 times larger when this last digit is carried to the beginning of the number.

87. All the 2−digit numbers from 19 to 93 are written consecutively to form the number N = $19202122 \ldots 919293$. Find the largest power of 3 that divides N.

88. If a, b, x and y are integers greater than 1 such that a and b have no common factors except 1 and $x^a = y^b$, show that $x = n^b$ and $y = n^a$ for integers n greater than 1.

89. Find all four – digit numbers having the following properties :

i. It is a square,

ii. Its first two digits are equal to each other and

iii. Its last two digits are equal to each other.

90. Determine with proof, all the positive integers n for which

i. n is not the square of any integer and

ii. $[\sqrt{n}]^3$ $divides$ n^2.

([x] denotes the largest integer that is less than or equal to x).

91. For a positive integer n, define A(n) to be $(2n)!/(n!)^2$. Determine the sets of positive integers n for which:

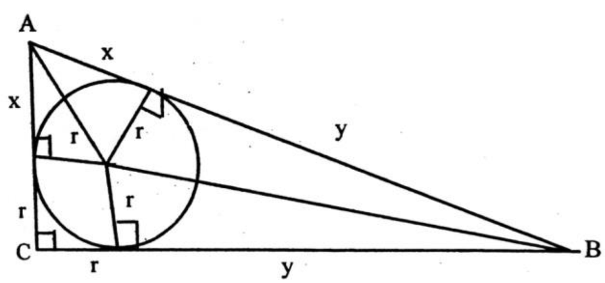
(i) A(n) is an even number; (ii) A(n) is a multiple of 4.

92. Given any positive integer n show that there are two positive rational numbers a and b, a ≠ b, which are not integers and

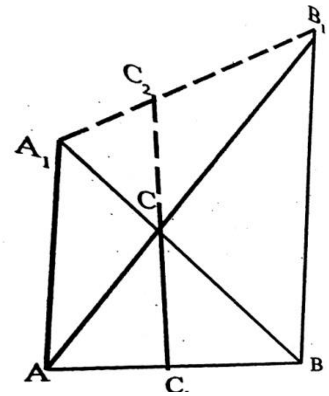
which are such that a –b, $a^2 - b^2, \dots, a^n - b^n$ are all integers.

Geometry

93. **Suppose ABCD is a cyclic quadrilateral. The diagonals AC and BD intersect at P. Let O be the circumcentre of triangle APB and H, the orthocenter of triangle CPD. Show that O, P, H are collinear.**
94. **In a triangle ABC, AB = AC. A circle is drawn touching the circumcircle of ΔABC internally and also, touching the sides AB and AC at P and Q respectively. Prove that the midpoint of PQ is the in centre of triangle ABC.**
95. **ABC is a right angled triangle with $\angle C = 90°$. The centre and the radius of the inscribed circle is I and r. Show that $AI \times BI = \sqrt{2} \times AB \times r$.**



96. **Let C_1 be any point on side AB of a triangle ABC. Draw C_1C meeting AB at C_1. The lines through A and B parallel to CC_1 meet BC produced and AC produced at A_1 and B_1 respectively. Prove that $\frac{1}{AA_1} + \frac{1}{BB_1} = \frac{1}{CC_1}$**



97. **If $u = cot\ 22°\ 30', v = \frac{1}{sin\ 22°30'}$, prove that u satisfies a quadratic and v satisfies a quartic (biquadratic or 4th degree) equation with integral coefficients which is a monic polynomial equation (i.e., the leading coefficient = 1).**

98. **Let AB and CD be two perpendicular chords of a circle with centre O and radius r and let X, Y, Z, W denote in cyclical order the four parts into which the disc is thus divided. Find the maximum and minimum of the quantity $\frac{E(X)+E(Z)}{E(Y)+E(W)}$, where E(u) denotes the area of u.**
99. **Two given circles intersect in two points P and Q. Show how to construct a segment AB passing through P and terminating on the two circles such that AP. PB is a maximum.**

100. **Let A, B, C, D be four given points on a line l. Construct a square such that two of its parallel sides or their extensions go through A and B respectively and the other two sides (or their extensions) go through C and D respectively.**

101. **The diagonals AC, BD of the quadrilateral ABCD intersect at the interior point O. The areas of the triangles AOB and COD are s_1 and s_2 respectively and the area**

of the quadrilateral is s. Prove that $\sqrt{s_1} + \sqrt{s_2} \leq \sqrt{s}$. When does equality hold?

102. Let M be the midpoint of the side AB of ΔABC. Let P be a point on AB between A and M and let MD be drawn parallel to PC, intersecting BC at D. If the ratio of the area of ΔBPD to that of ΔABC is denoted by r, then examine which of the following is true:

(a) $\frac{1}{2} < r < 1$ depending upon the position of P.

(b) $r = \frac{1}{2}$

(c) $\frac{1}{3} < r < \frac{2}{3}$ depending upon the position of P.

103. ABCDE is a convex pentagon inscribed in a circle of radius 1 units with AE as diameter. If AB $= a, BC = b$, CD $= c, DE = d$, prove that $a^2 + b^2 + c^2 + abc + bcd < 4$.

104. A rhombus has half the area of the square with the same side length. Find the ratio of the longer diagonal to that of the shorter one.

105. A ball of diameter 13 cm is floating so that the top of the ball is 4cm above the smooth surface of the pond. What is the circumference in centimeters of the circle formed by the contact of the water surface with the ball.

106. OPQ is a quadrant of a circle and semicircles are drawn on OP and OQ. Show that the shaded areas a and b are equal.

107. Given a circle of radius 1 unit and AB is a chord of the circle with length 1 unit. If C is any point on the major segment, show that $AC^2 + BC^2 \leq 2(2 + \sqrt{3})$.

108. From a point E on the median AD of ΔABC, the perpendicular EF is dropped to the side BC. From a point M on EF, perpendiculars MN and MP are drawn to the sides AC and AB respectively. If N, E, P are collinear, show that M lies on the internal bisector of $\angle BAC$.

109. AD is the internal bisector of $\angle A$ in ΔABC. Show that the line through D drawn parallel to the tangent to the circumcircle at A touches the inscribed circle.

110. Given two concentric circles of radii R and r. From a point P on the smaller circle, a straight line is drawn to intersect the larger circle at B and C. The perpendicular to BC at P intersects the smaller circle at A. Show that

$$PA^2 + PB^2 + PC^2 = 2(R^2 + r^2).$$

111. Find x, y, z $\in R$ satisfying $\frac{4\sqrt{x^2+1}}{x} = \frac{5\sqrt{y^2+1}}{y} = \frac{6\sqrt{z^2+1}}{z}$ and xyz $= x + y + z$.

112. If $a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x = 0$ for all $x \in R$, show that $a_0 = a_1 = a_2 = a_3 = 0$.

113. If any straight line is drawn cutting three concurrent lines OA, OB, OP at A, B, P, then

$$\frac{AP}{PB} = \frac{AO \sin AOP}{BO \sin POB}$$

114. ABC is a triangle that is inscribed in a circle. The angle bisectors of A, B, C meet the circle at D, E, F respectively. Show that AD is perpendicular to EF.

115. ABC is a triangle. The bisectors of $\angle B\ and\ \angle C$ meet AC and AB at D and E respectively and BD and CE intersect at O. If OD=OE, prove that either $\angle BAC = 60°$ or the triangle is isosceles.

116. Show that the radian measure of an acute angle is less than harmonic mean of its sine and its tangent.

117. Show how to construct a chord BPC in a given angle A, through a given point P, such that $\frac{1}{BP}+\frac{1}{PC}$ is maximum, where P is in the interior of $\angle A$.

118. If a line AQ of an equilateral triangle ABC, is extended to meet the circumcircle at P, then $\frac{1}{PB}+\frac{1}{PC}=\frac{1}{PQ}$ where Q is the point where AQ meets BC.

119. Let ABC be a triangle of area Δ and A'B'C' be the triangle formed by the altitudes $h_a, h_b, h_c\ of\ \Delta ABC$ as its sides with area $\Delta'$and A''B''C'' be the triangle formed by the altitudes of ΔA'B'C' as its sides with area Δ''. If $\Delta' = 30\ and\ \Delta'' = 20$, find D.

120. Let ABC be a right angled triangle right angled at A and S be its circumcircle. Let S_1 be the circle touching AB, AC and circle S internally. Let S_2 be the circle touching AB, AC and S externally. If $r_1\ and\ r_2$ are the radii of circles S_1 and S_2 respectively, show that $r_1.r_2 = 4\ area\ (\Delta ABC)$.

121. Two circles C_1 and C_2 intersect at two distinct points P and Q in a plane. Let a line passing through P meet the circles C_1 and C_2 in A and B respectively. Let Y be the midpoint of AB and QY meet the circles C_1 and C_2 in X and Z respectively. Show that Y is also the midpoint of XZ.

122. Given a triangle ABC in a plane Σ find the set all points P lying in the plane Σ such that the circumcircles of triangles ABP, BCP and CAP are congruent.

123. Suppose ABCD is a convex quadrilateral and P.Q are the midpoints of CD, AB. Let AP, DQ meet in X and BP, CQ meet in Y. Prove that [ADX] + [BCY] = [PXQY]. How does the conclusion alter if ABCD is not a convex quadrilateral?

124. A triangle ABC has in centre I. Points X, Y are located on the line segments AB, AC respectively so that BX. AB $= IB^2$ and CY.AC $= IC^2$. Given that X, I, Y are collinear, find the possible values of the measure of angle A.

125. Suppose $A_1A_2A_3 \dots A_n$ is an n−sided regular polygon such that

$$\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4}$$

Determine n, the number of sides of the polygon.

126. Let ABC be a triangle with $\angle A = 90°$, and S be its circumcircle. Let S_1 be the circle touching the rays AB, AC and the circle S internally. Further let S_2 be the circle touching the rays AB, AC and the circle S externally. If r_1, r_2 be the radii of

the circles S_1 and S_2 respectively, show that $r_1r_2 = 4[ABC]$.

127. Suppose ABCD is a rectangle and P, Q, R, S are points on the sides AB, BC, CD, DA respectively. Show that PQ + QR + RS + SP $\geq \sqrt{2}$ AC.

128. Let ABC be a triangle and h_a be the altitude through A. Prove that

$$(b + c)^2 \geq a^2 + 4{h_a}^2$$

(As usual a, b, c denote the sides BC, CA, AB respectively).

129. Let P be an interior point of a triangle ABC and let BP and CP meet AC and AB in E and F respectively. If [BPF] $= 4$, $[BPC] = 8$ and $[CPE] = 13$, find [AFPE]. (Here [] denotes the area of a triangle or a quadrilateral as the case may be).

130. Suppose ABCD is a cyclic quadrilateral inscribed in a circle of radius one unit.

If AB. BC. CD. DA ≥ 4. Prove that ABCD is a square.

COMBINATORICS

131. Find the number of ways to choose an ordered pair (a, b) of numbers from the set (1, 2, …, 10) such that |a−b| $\leq$ 5.

132. Identify the set S by the following information :

(i) $S \cap \{3, 5, 8, 11\} = \{5, 8\}$

(ii) $S \cup \{4, 5, 11, 13\} = \{4, 5, 7, 8, 11, 13\}$

(iii) $\{8, 13\} \subset S$

(iv) $S \subset \{5, 7, 8, 9, 11, 13\}$

Also show that no three of the conditions suffice to identify S uniquely.

133. Let S be the set of pensioners, E the set of those that lost an eye, H those that lost an ear, A those that lost an arm and L those that lost a leg. Given that n(E) $= 70\%$, n (H) < 75%, $n(A) = 80\%$ and $n(L) = 85\%$, find what percentage at least must have lost all the four.

134. Let set S $= \{a_1, a_2, a_3, \ldots, a_{12}\}$ where all twelve elements are distinct, we want to form sets each of which contains one or more of the elements of set S (including the possibility of using all the elements of S). The only restriction is that the subscript of each element in a specific set must be an integral multiple of the smallest subscript in the set. For example $\{a_2, a_6, a_8\}$ is one acceptable set, as is $\{a_6\}$. How many such sets can be formed? Can you generalize the result?

135. Prove that there are $2(2^{n-1} - 1)$ ways of dealing n cards to two persons. (The players may receive unequal numbers of cards).

136. Let S be the set of natural numbers whose digits are chosen from {1, 2, 3, 4} such that

(a) When no digits are repeated, find n(S) and the sum of all numbers in S.

(b) When S_1 is the set of upto 4−digit numbers where digits are repeated. Find $|S_1|$ and also find the sum of all the numbers in S_1.

137. Find the number of 6 digit natural numbers where each digit appears at least twice.

138. **Let $X = \{1, 2, 3, \ldots N\}$ $where\ n \in N$. Show that the number of r combinations of X which contain no consecutive integers is given by $\binom{n-r+I}{r}$ where $0 \leq r \leq n-r+I$.**

139. **Let S = {1, 2, 3, …, (n+1), where n $\geq$ 2 and let t $= \{(x, y, z) | x, y, z \in S, x < y, y < z\}$. By counting the members of T in two different ways, prove that**

$$\sum_{k=1}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$$

140. **Find the number of permutations**

(p_1, p_2, p_3, p_4, p_5, p_6) of (1, 2, 3, 4, 5, 6) such that for any k, $1 \leq k \leq 5$ (p_1, p_2, p_3, …, p_k) does not form a permutation of 1, 2, 3, …, k, i.e., p_1 $\neq$ 1, (p_1, p_2) is not a permutation of (1, 2) (p_1, p_2, p_3) is not a permutation of (1, 2, 3), etc.

141. **Consider the collection of all three element subsets drawn from the set {1, 2, 3, 4, …, 299, 300}. Determine the number of subsets for which, the sum of the elements is a multiple of 3.**

142. **4^n + 1 points lie within an equilateral triangle of side 1 com. Show that it is possible to choose out of them, at least two, such that the distance between them is $\frac{1}{2^n}$ cm.**

143. **Let A be any set of 19 distinct integers chosen from the Arithmetic Progression 1, 4, 7, …, 100. Prove that there must be two distinct integers in A, whose sum is 104.**

144. **Let X $\subset$ {1, 2, 3, …99} and n(X) = 10. Show that it is possible to choose two disjoint non−empty proper subsets Y, Z of X such that $\sum(y | y \in Y) = \sum(z | z \in Z)$.**

145. **Find the number of integer solutions to the equation**
$x_1 + x_2 + x_3 = 28\ where\ 3 \leq x_1 \leq 9, 0 \leq x_2 \leq 8\ and\ 7 \leq x_3 \leq 17$.

146. **I have six friends and during a certain vacation, I meet them during several dinners. I found that I dined with all the six exactly on one day, with every five of them on 2days, with every four of them on 3 days, with every three of them on 4 days and with every two of them on 5 days. Further every friend was present at 7 dinners and every friend was absent at 7 dinners. How many dinners did I have alone?**

147. **Let A denote the subset of the set $S =$ {a, a+d, …, a+2nd} having the property that no two distinct elements of A add up to 2(a+ nd). Prove that A cannot have more than (n+1) elements. If in the set S, a+ 2nd is changed to a + (2n +1)d, what is the maximum number of elements in A if in this case no two elements of A add up to 2a + (2n + 1)d?**

148. **Show that the number of three elementic subsets (a, b, c) of {1, 2, 3, …, 63} with (a+b+c) < 95 is less than the number of those with (a + b + c) > 95.**

149. **Given any five distinct real numbers, prove that there are two of them, say x and y, such that**

0 < (x – y)/(1 + xy).

150. **Show that using $1 = 3^0, 3^1, 3^2, \dots, 3^n$, weight, i.e., (n + 1) weight each of which is of the form $3^i, 0 \le i \le n$, one can weight all the objects weighing from 1 unit to**

$$1 + 3 + 3^2 + \cdots + 3^n = \frac{3^{n+1} - 1}{2} \text{ units}.$$

151. **To cross a river there is a boat which can hold just two persons. n newly wedded couples want to cross the river to reach the far side of the river. But husbands and wives have no mutual confidence in the other. So, none of them want to leave his (her) wife (husband) along with other man (woman). But they do not mind leaving them alone or with at least one more couple. How many times they have to row front and back so that all the couples reach the famside of the river?**

152. **A difficult mathematical competition consisted of a Part I and a part II with a combined total of 28 problems. Each contestant solved 7 problems altogether. For each pair of problems there were exactly two contestants who solved both of them. Prove that there was a contestant who in Part I solved either no problem or at least 4 problems.**

153. **It is proposed to partition the set of positive integers into two disjoint subsets A and B. Subject to the following conditions:**

(i) **I is in A.**

(ii) **No two distinct members of A have a sum of the form $2^k + 2 (k = 0, 1, 2, \dots)$.**

(iii) **No two distinct members of B have a sum of the form $2^k + 2 (k = 0, 1, 2, \dots)$.**

Show that this partitioning can be carried out in a unique manner and determine the subsets to which 1987, 1988, 1989, 1997, 1998 belong.

154. **Suppose $A_1, A_2, \dots, A_6$ are six sets each with four elements and $B_1, B_2, \dots, B_n$ are n sets each two elements such that**

$$A_1 \cup A_2 \cup \dots \cup A_6 = B_1 \cup B_2 \cup \dots \cup B_n = S \text{ (say)}$$

Given that each element of S belongs to exactly four of the A_i's and exactly three of the B_j' s, find n.

155. **Two boxes contain between 65 balls of several different sizes. Each ball is white, black, red, or yellow. If you take any five balls of the same colour, at least two of them will always be of the same size (radius). Prove that there are at least three balls which lie in the same box, have the same colour and are of the same size.**

156. **Let A denote a subset of the set {1, 11, 21, 31, …, 541, 551} having the property that no two elements of A add up to 552. Prove that A cannot have more than 28 elements.**

157. **Find the number of permutations, $(P_1, P_2, \dots, P_6)$ of (1, 2, …, 6) such that for any k, $1 \le k \le 5, (P_1, P_2, \dots, P_k)$ does not form a permutation of 1, 2, …, k. [That is $P_1 \neq 1; (P_1, P_2)$ is not a permutation of 1, 2, 3, etc.]**

158. **Let A be a subset of {1, 2, 3, ….2n−1, 2n} containing n+1 elements. Show that**

a. **Some two elements of A are relatively prime:**

b. **Some two elements of A have the property that one divides the other.**

159. **Let A denote the set of all numbers between 1 and 700 which are divisible by 3 and let B denote the set of all numbers between 1 and 300 which are divisible by 7. Find the number of all ordered pairs (a, b) such that $a \in A, b \in B, a \neq b$ and a + b is even.**

160. **Find the number of unordered pairs {A, B} (i.e., the pairs {A, B} and {B, A} are considered to be the same) of subsets of an n−element set X which satisfy the conditions :**

(a) $A \neq B$;

(b) $A \cup B = X$.

161. **Find the number of quadratic polynomials, ax^2+bx+c, which satisfy the following conditions :**

(a) a, b, c are distinct;

(b) $a, b, c \in \{1, 2, 3, \dots, 1999\}$ **and**

(c) $x + 1$ **divides** ax^2+bx+c.

162. **Let X be a set containing n elements. Find the number of all ordered triplets (A, B, C) of subsets of X such that A is a subset of B and B is a proper subset of C.**

163. **Find the number of** 4×4 **arrays whose entries are from the set {0, 1, 2, 3} and which are such that the sum of the numbers in each of the four rows and in each of the four columns is divisible by 4. (An** $m \times n$ **array is an arrangement of mn numbers in m rows and n columns).**

164. **There is a** $2n \times 2n$ **array (matrix) consisting of 0's and I's there are exactly 3n zeroes. Show that it is possible to remove all the zeroes by deleting some n rows and some n columns.**

165. **For any natural number n, (n $\geq$3), let f(n) denote the number of non−congruent integer−sided triangle with perimeter n (e.g., f(3)** $= 1, f(4) = 0, f(7) = 2$**). Show that**

(a) f(1999) > f(1996);

(b) f(2000) = f(1997)

SOLUTION SET

Algebra

1. $(a-b)^2 \geq 0$

$\Rightarrow a^2 + b^2 \geq 2ab$

$\Rightarrow \frac{a}{b} + \frac{b}{a} \geq 2$

$a_1 + a_2 + a_3 + \cdots + a_n = 1$ ....(1)

Dividing equation (1) by $a_1, a_2, a_3, \ldots, a_n$ successively and adding, we get

$$1 + \frac{a_2}{a_1} + \frac{a_3}{a_1} + \cdots + \frac{a_n}{a_1} = \frac{1}{a_1}$$

$$\frac{a_1}{a_2} + 1 + \frac{a_3}{a_2} + \cdots + \frac{a_n}{a_2} = \frac{1}{a_2}$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$\frac{a_1}{a_r} + \frac{a_2}{a_r} + \cdots + \frac{a_{r-1}}{a_r} + 1 + \frac{a_{r+1}}{a_r} + \cdots + \frac{a_n}{a_r} = \frac{1}{a_r}$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$\frac{a_1}{a_n} + \frac{a_2}{a_n} + \frac{a_3}{a_n} + \cdots + \frac{a_{n-1}}{a_n} + 1 = 1/a_n$$

Adding (1 + 1 + n terms) $= \sum \frac{a_i}{a_j}; i \neq j, i, j = 1, 2, 3, \ldots n$

$$\sum_{i=1}^{n} \frac{1}{a_i}$$

In $\sum \frac{a_i}{a_j}$ there are n(n−1) fractions. $\frac{a_i}{a_j}$ are all distinct. Pairing $\frac{a_i}{a_j}$ and $\frac{a_j}{a_i}$ there are $\frac{n(n-1)}{2}$ pairs of fractions of the form $\frac{a_i}{a_j} + \frac{a_j}{a_i}$.

But each $\frac{a_i}{a_j} + \frac{a_j}{a_i} \geq 2$

$$\therefore \sum_{i=1}^{n} \frac{1}{a_i} \geq n + \frac{n(n-1)}{2} \times 2$$

$$\sum_{i=1}^{n} \frac{1}{a_i} \geq n + n^2 - n = n^2$$

Equality holds when all a_i are equal, i.e., each is equal to $\frac{1}{n}$.

Aliter : By A.M. –G.M. inequality

$$\frac{\sum a_i}{n} \geq (a_1 + \cdots + a_n)^{1/n}$$

$$\frac{\sum \frac{1}{a_i}}{n} \geq \left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right)^{1/n}$$

Since both the sides of the inequalities are positive, we have

$$\frac{\sum a_i}{n} . \frac{\sum \frac{1}{a_i}}{n} \geq 1$$

Since $\sum a_i = 1$, we get $\sum \frac{1}{a_i} \geq n^2$.

2. By A.M. –G.M. inequality,

$$\sqrt{a_1 a_2} \leq \frac{a_1 + a_2}{2}$$

$$\sqrt{a_1 a_3} \leq \frac{a_1 + a_3}{2}$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

… … … … … … … … … …

$$\sqrt{a_{n-1}a_n} \leq \frac{a_{n-1}+a_n}{2}$$

Where $i \neq j, i, j = 1, 2, \ldots n$

There are $\frac{n(n-1)}{2}$ inequalities and, on the right hand side, each a_i occurs (n−1) times.

Adding these inequalities, we get

$$\sqrt{a_1a_2} + \sqrt{a_1a_3} + \cdots + \sqrt{a_ia_j} + \cdots + \sqrt{a_{n-1}a_n} \leq (n-1)$$

$$\left(\frac{a_1+a_2+\cdots+a_n}{2}\right) = \frac{n-1}{2}(a_1+a_2+\cdots+a_n)$$

3. Applying Cauchy Schwarz inequality for w^2, x^2, y^2, z^2 and w, x, y, z, we get

$(w^3+x^3+y^3+z^3)^2 \leq (w^4+x^4+y^4+z^4)(w^2+x^2+y^2+z^2)$…….(1)

Again applying Cauchy Schwarz inequality with w^2, x^2, y^2, z^2 and 1, 1, 1, 1, we get

$(w^2+x^2+y^2+z^2)^2 \leq (w^4+x^4+y^4+z^4)^4$

$\Rightarrow (w^2+x^2+y^2+z^2) \leq (w^4+x^4+y^4+z^4)^2$ ……….(2)

$\therefore (w^4+x^4+y^4+z^4) \geq \frac{(w^3+x^3+y^3+z^3)^2}{(w^2+x^2+y^2+z^2)}$, by Eq. (1)

$\Rightarrow \qquad \geq \frac{(w^3+x^3+y^3+z^3)^2}{2(w^4+x^4+y^4+z^4)^{1/2}}$ by (2)

$\Rightarrow (w^4+x^4+y^4+z^4)^{3/2} \geq \frac{100}{2} = 50$

$\Rightarrow w^4+x^4+y^4+z^4 \geq 50^{\frac{2}{3}} or \sqrt[3]{2500}$

4. Here x > 0 and $x \neq 1$.

Let $log_2x = p \; as \; x \neq 1, p \neq 0$.

The given inequality becomes $p + \frac{1}{p} + 2\cos y \leq 0$

$i.e., \frac{p^2+1+2p\cos y}{p} \leq 0.$

Case I: when p > 0

$p^2+1+2p\cos y \leq 0 \; for \; all \; y \; and \; p > 0$

$(p-1)^2 + 2p(1+cosy) \leq 0$ ………(1)

Since p > 0 therefore 1 + cos y ≥ 0, $(p-1)^2 \geq 0$

$(p-1)^2 + 2p(1+cosy) \geq 0$ ……,(2)

The only way both equation (1) and equation (2) are satisfied when

$(p-1)^2 + 2p(1+\cos y) = 0$

Since, $(p-1)^2 \geq 0 \; and \; 2p(1+\cos y) \geq 0$, we get

$(p-1)^2 = 0 \; and \; 2p(1+\cos y) = 0$

$\therefore p = 1 \; and \cos y = -1$

$\therefore y = (2n+1)\pi$

Solution set is x =2 and y = $(2n+1)\pi$.

Case II: when p < 0

$p^2+1+2pcos\, y \geq 0$

$(p+1)^2 - 2p(1-\cos y) \geq 0$

$(p+1)^2 \geq 0 \; and -p(1-\cos y) \geq 0$ for all y

∴ solution set is 0<x<1 and all y ∈ R.

5. Clearly P(n) × P(n + 1) is of fourth degree in 'n' as P(n) and P(n+1) are of second degree each in n, and so P(n) × P(n+1) will be a polynomial of 4th degree in n, with leading coefficient 1.

So, if there exists an M so that P(M) = $P(n) \times P(n+1)$, then M must be in the form of quadratic in n, with leading coefficient 1.

Let M $= n^2 + cn + d$, where c and d are integers.
Now P(M) $= P(n^2 + cn + d)$
$= (n^2 + cn + d)^2 + a(n^2 + cn + d) + b$
$= n^4 + 2cn^3 + (c^2 + 2d + a)n^2 + (2cd + ac)n + d^2 + ad + b$

$and\ P(n) \times P(n+1) = (n^2 + an + b)[(n+1)^2 + a(n+1) + b]$
$= n^4 + 2(a+1)n^3 + [(a+1)^2 + (a+2b)]n^2 + (a+1)(a+2b)n + b(a+b+1)$

Now comparing, the coefficients of n^3 constant terms of P(M) and P(n) × $P(n+1)$, we get
$2c = 2(a+1)$
$\Rightarrow c = (a+1)$
And $d^2 + ad + b \ = ab + b^2 + b$
$\Rightarrow d^2 - b^2 + ad - ab = (d-b)(d+a+b) = 0$
$\Rightarrow d = b\ or\ d = -(a+b)$

Using these values of $d = b\ and\ c = a + 1$, the coefficient of n^2 and n in P(M) are

$c^2 + 2d + a = (a+1)^2 + 2b + a$
And 2cd+ac = 2 (a+1) b+ a(a+1)
$= (a+1)(2b+a)$ respectively

But these are the coefficients of n^2 and n in P(n) × $P(n+1)$. Thus, with these values for c and d, P(M) $= P(n) \times P(n+1)$.

So, the M of the desired property is $n^2+(a+1)n+b$.

Thus, we can verify that $d = -(a+b), c = (a+1), if\ P(M) and\ P(n) \times P(n+1)$ are identical and, hence, show that there exists exactly one M for every n which is a function of n, i.e.

$$M = f(n) = n^2 + (a+1)n + b$$

6. Let P= $x^{9999} + x^{8888} + x^{7777} + \ldots + x^{1111} + 1$

And Q= $x^9 + x^8 + x^7 + \ldots + x + 1$

$P - Q = x^9(x^{9990} - 1) + x^8(x^{8880} - 1) + x^7(x^{7770} - 1) + \cdots + x(x^{1110} - 1)$

$= x^9[(x^{10})^{999} - 1] + x^8[(x^{10})^{888} - 1] + x^7[(x^{10})^{777} - 1] + \cdots + x[(x^{10})^{111} - 1]$ ……(1)

But $(x^{10})^n \ - 1$ is divisible by $x^{10} - 1\ for\ all\ n \geq 1$.

$\therefore R.H.S.$ of equation (1) is divisible by $x^{10} - 1$.

$\therefore P - Q$ is divisible by $x^{10} - 1$ and hence divisible by $x^9 + x^8 + \ldots + 1$.

7. We shall use the identity

$$a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

Writing a$= x, b = 5^{1/3}y, c = 5^{2/3}z$ in the given equation, it can be written as

$x^3 + (5^{\frac{1}{3}}y)^3 + (5^{\frac{2}{3}}z)^3 - 3 \times x \times 5^{1/3}y \times 5^{2/3}z = 0$

$$\therefore \frac{1}{2}\left(x + 5^{\frac{1}{3}}y + 5^{\frac{2}{3}}y\right)\left[\left(x - 5^{\frac{1}{3}}y\right)^2 + \left(5^{\frac{1}{3}}y - 5^{\frac{2}{3}}z\right)^2 + \left(5^{\frac{2}{3}}z - x\right)^2\right]$$

$$\Rightarrow (x + 5^{\frac{1}{3}}y + 5^{\frac{2}{3}}z) = 0$$

$$or \left[\left(x - 5^{\frac{1}{3}}y\right)^2 + \left(5^{\frac{1}{3}}y - 5^{\frac{2}{3}}y\right)^2 + \left(5^{\frac{2}{3}}z - x\right)^2\right] = 0$$

If $x + 5^{1/3}y + 5^{2/3}z = 0, then\ 5^{1/3}y + 5^{2/3}z = -x$

Clearly the left hand side is irrational, when y and z are integers other than zero and the right hand side is also an integer.

So $x = y = z = 0$ is a solution.

If $\left(x - 5^{\frac{1}{3}}y\right)^2 + \left(5^{\frac{1}{3}}y - 5^{\frac{2}{3}}z\right)^2 + \left(5^{\frac{2}{3}}z - x\right)^2 = 0, then$

$x = 5^{1/3}y, y = 5^{1/3}z\ and\ x = 5^{2/3}z.$

Again this is possible only when $x = y = z = 0$ as we need integer values for x, y, and z.

Aliter : number theoretic solution

$x^3 + 5y^3 + 25z^3 - 15xyz = 0$

$\Rightarrow x^3 = 5(3xyz - y^3 - 5z^3)$ ……….(1)

$\Rightarrow \frac{5}{x^3}$ and hence 5/x.

Let $x = 5x_1\ then\ x^3 = 125{x_1}^3$

So that the equation becomes

$y^2 = 5x_1yz - 25{x_1}^3 - 5z^3$

$\Rightarrow \frac{5}{y}\ and\ let\ y = 5y_1$

Again the equation becomes $z^3 = 15zx_1y_1 - 5{x_1}^3 - 25{y_1}^3$

$\Rightarrow \frac{5}{z}\ and\ taking\ z = 5z_1$

We get ${x_1}^3 + 5{y_1}^3 + 25{z_1}^3 - 15x_1y_1z_1 = 0$ ……….(2)

This implies that if (x, y, z) is an integral solution, then (x/5, y/5, z/5) is also an integral solution to eq. (1).

Arguing in the same way we find

$$x_2 = \frac{x_1}{5}, y_2 = \frac{y_1}{5}, z_2 = \frac{z_1}{5}$$

$$or, x_2 = \frac{x}{5^2}, y_2 = \frac{y}{5^2}, z_2 = \frac{z}{5^2}$$

Is also an integral solution and thus, by induction method, we get

$x_n = \frac{x}{5^n}, y_n = \frac{y}{5^n}, z_n = \frac{z}{5^n}$

Is an integral solution for all $n \geq 0$.

This mean that x, y and z are multiple of 5^n, for all $n \in N$.

This is possible only when x, y, z are all zero.

8. Suppose $log_a x\ = b$

Then $x = a^b = (a^n)^{\frac{b}{n}}$

$\Rightarrow log_{a}{}^{n}x = \frac{b}{n}$

$\Rightarrow n\ log_{a^n}\ x = b$

$\Rightarrow log_{a^n} x^n = b = log_a x$

$so\ log_2 x = log_{2^2} x^2 = log_4 x^2$

$log_3 y = log_{3^2} y^2 = log_9 y^2$

$log_4 z = log_{4^2} z^2 = log_{16} y^2$

$so\ log_2 x + log_3 y + log_4 z = 2$

$\Rightarrow log_4 x^2yz = 2$

$\Rightarrow x^2yz = 4^2 = 16$ ………(1)

Similarly, $y^2xz = 81$ ……….(2)

And $z^2xy = 256$ ……….(3)

And hence $x^2yz \times y^2xz \times z^2xy = 16 \times 81 \times 256$

$\Rightarrow (xyz)^4 = 2^4 \times 3^4 \times 4^4$

xyz = 24 as x, y, z > 0

Dividing eq. (1), (2), and (3) by xyz = 24, we can get

$$x = \frac{16}{24}, y = \frac{81}{24}, z = \frac{256}{24}$$

$$\Rightarrow x = \frac{2}{3}, y = \frac{27}{8}, z = \frac{32}{3}$$

9. If possible, let us express

$x^4 + 26x^3 + 52x^2 + 78x + 1989 \equiv (x^2 + ax + b)(x^2 + cx + d)$

By comparing coefficients of both sides, we get

a + c = 26……….(1)

ac+b+d= 52………..(2)

bc + ad= 78………(3)

bd = $1989 = 13 \times 3^2 \times 17$ ………..(4)

Now, we see that 13 is a divisor of 26, 52, 78 and 1989 and 13 is a prime number.

Thus, 13/ b.d.

⇒ 13 divides one of b or d but not both.

If 13/b say and 13'/d, them from eq. (3), 13/a.

Now, 13/ac and 13/b, and 13/52.

∴ 13/d, from (2) is a contradiction.

Such a factorization is not prime.

∴ 13'/d, it is a contradiction.

So if 13/d and 13/b, then again from eq. (3), 13/c. [From eq. (1) 13/a also].

Now, $b = 52 - ac - d.$

13/b, 13/52, 13/ac, 13/d but it is again a contradiction.

So there does not exist quadratic polynomials p(x) and q(x) with integral coefficients such that f(x) = $p(x) \times q(x)$.

Similarly, if p(x) is a cubic polynomial and q(x) is a linear one then let

$p(x) = x^3 + ax^2 + bx + c$

$q(x) = (x + d)$

$x^4 + 26x^3 + 52x^2 + 78x + 13 \times 3^2 \times 17 = (x^3 + ax^2 + bx + c)(x + d)$

Again comparing coefficients

$$a + d = 26 \dots \dots (5)$$

$$ad + b = 52 \dots \dots \dots .. (6)$$

$$bd + c = 78 \dots \dots \dots . (7)$$

$$cd = 13 \times 3^2 \times 17 \dots \dots \dots (8)$$

As before 13 divides exactly one of c and d.

If 13/d, then 13'/c, then by eq. (7), 13/bd, 13'/c and 13/78 $= bd + c$ is a contradiction.

So let 13/c, 13'/d.

By eq. (7), 13/b,

By eq. (6), 13/b and 13/52

$\Rightarrow 13/ad$

$\Rightarrow 13/a$, as 13/d.

By eq. (5), 13/a, 13/d and 13/26 $= a + d$ (a contradiction) and hence there does not exist any polynomials p(x) and q(x) as assumed and hence the result.

10. x, y, z are integers and 5 is a prime number and given equation is

$$x^{y^z}.y^{z^x}.z^{x^y} = 5xyz$$

Dividing both sides of the equation by xyz,

$$x^{y^z-1}.y^{z^x-1}.z^{x^y-1} = 5$$

So the different possibilities are

$$\left.\begin{matrix} x^{y^z-1} = 5 \\ y^{z^x-1} = 1 \\ z^{x^y-1} = 1 \end{matrix}\right| \; or \; \left.\begin{matrix} x^{y^z-1} = 1 \\ y^{z^x-1} = 5 \\ z^{x^y-1} = 1 \end{matrix}\right| \; or \; \left.\begin{matrix} x^{y^z-1} = 1 \\ y^{z^x-1} = 1 \\ z^{x^y-1} = 5 \end{matrix}\right|$$

Taking the first column

$x = 5, y^2 - 1 = 1;\ y^z = 2, y = 2, and\ z = 1$

And these values are satisfying the other expressions in the first column.

Similarly from the second column, we get $y = 5, z = 2, x = 1$ and from the third column, we get $z = 5, x = 2, y = 1$.

11. If $k = 1$

$P_1(x) = x^9 + x^3 + x^2 + x + 1 = x^9 - x^4 + x^4 + x^3 + x^2 + x + 1 = x^4(x^5 - 1) + (x^4 + x^3 + x^2 + x + 1) = x^4(x - 1)(x^4 + x^3 + x^2 + x + 1) + (x^4 + x^3 + x^2 + x + 1) = (x^4 + x^3 + x^2 + x + 1)[x^4(x - 1) + 1]$

Thus, $x^4 + x^3 + x^2 + x + 1$ is a non–trivial polynomial divisor of $P_1(x)$.

$P_k(x) = x^{(5k+4)} - x^4 + (x^4 + x^3 + x^2 + x + 1) = x^4[x^{5k} - 1] + (x^4 + x^3 + x^2 + x + 1)$

$(x^5 - 1)$ divides $(x^5)^k - 1, x^4 + x^3 + x^2 + x + 1$ divides $x^5 - 1\ and\ hence\ x^{5k} - 1.$

Therefore $x^4 + x^3 + x^2 + x + 1\ divides\ P_k(x)$ for all k.

12. Suppose there exists an integer b such that f(b) = 1993

Let $g(x) = f(x) - 1991.$

Now, g is a polynomial with integer coefficients and

$g(a_i) = 0 for\ i = 1,2,3,4$

Thus $(x - a_1)(x - a_2)(x - a_3) and\ (x - a_4)$ are all factors of g(x).

So, $g(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4) \times h(x)$

Where h(x) is polynomial with integer coefficients.

$g(b) = f(b) - 1991$

$= 1993 - 1991 = 2$ (by our choice of b)

But $g(b) = (b - a_1)(b - a_2)(b - a_3)(b - a_4)h(b) = 2$

Thus, $(b-a_1)(b-a_2)(b-a_3)(b-a_4)$ are all divisors of 2 and are distinct.

$\therefore (b-a_1)(b-a_2)(b-a_3)(b-a_4) are$ $1, -1, 2, 2, -2$ in some order and h(b) is an integer.

$\therefore g(b) = 4.h(b) \neq 2.$

Hence such a and b does not exist.

13. Let x, y, z be the roots of the cubic equation

$t^3 - at^2 + bt - c = 0$ ....(1)

$s_1 = x + y + z = a$ ....(2)

$s_2 = xy + yz + zx = b$ ....(3)

$\Rightarrow 2xy + 2yz + 2zx = 2b$ ....(4)

From eq. (2) we get $a = 3$.

From eq. (2) and eq. (3) we get

$2b = 2xy + 2yz + 2zx = (x + y + z)^2 - (x^2 + y^2 + z^2)$

$= 9 - 3 = 6$

$\Rightarrow b = 3$

Since x, y and z are the roots of eq. (1), substituting and adding, we get

$(x^3 + y^3 + z^3) - a(x^2 + y^2 + z^2) + b(x + y + z) - 3c = 0$

$\Rightarrow 3 - 3a + 3b - 3c = 0$

$\Rightarrow 3 - 9 + 9 - 3c = 0$

$\Rightarrow c = 1$

Thus eq. (1) becomes

$t^3 - 3t^2 + 3t - 1 = 0$

$\Rightarrow (t-1)^3 = 0$

Thus, the roots are 1, 1, 1.

Hence $x = y = z = 1$ is the only solution for the given equation.

14. $1 + x^n + x^{2n} + \cdots + x^{mn} = \frac{x^{(m+1)n}-1}{x^n-1}$ (verify)

$and\ 1 + x + x^2 + \cdots + x^m = \frac{x^{m+1}-1}{x-1}$

We must find m, and n so that $\frac{1+x^n+x^{2n}+\cdots+x^{mn}}{1+x+x^2+\cdots+x^m}$ is a polynomial

$\frac{x^{(m+1)n}-1}{x^n-1} \div \frac{x^{m+1}-1}{x-1} = \frac{[x^{(m+1)n}-1](x-1)}{(x^n-1)(x^{m+1}-1)}$ must be a polynomial.

Now, if k and l are relatively prime, then $(x^k - 1)$ and $(x^1 - 1)$ have just one common factor. For the roots of $x^k - 1 = 0$, say 1, w_1, w_2, ...w_{k-1} are all distinct factor.

Similarly, also those of $x^1 - 1 = 1, w'_1, w'_2, \ldots, w'_{r-1}$ is distinct factor.

By De Moivre's formula, the roots of $x^k - 1$ are

$$\cos\frac{2n\pi}{k} + i\ sin\frac{2n\pi}{k}$$

For n $= 0, 1, 2 \ldots, k-1$ and those of $x^1 - 1$ are $\cos\frac{2n\pi}{l} + i\ sin\frac{2n\pi}{l}$

For n= $0, 1, 2, \ldots l-1$. If l and k are prime integer other than zero,

Roots will be $\cos\frac{2n\pi}{l} + i\ sin\frac{2n\pi}{l}\ and \cos\frac{2n\pi}{k} + i \sin\frac{2n\pi}{k}$ respectively.

Since all the factors of $x^{n(m+1)} - 1$ are distinct, $x^{m+1} - 1, x^n - 1$ cannot have any common factors other than (x−1). Thus (m+1) and n must be relatively prime.

Again $x^{n(m+1)} - 1 = (x^n)^{m+1} - 1 = (x^{m+1})^n - 1.$

So $x^{n(m+1)} - 1$ is divisible by $(x^n - 1)$ and also by $(x^{m+1}) - 1.$

$$thus, \frac{\left[x^{(m+1)n} - 1\right](x-1)}{(x^n - 1)(x^{m+1} - 1)}$$

Is a polynomial which shows that the condition (m + 1) and n must be relatively prime is also sufficient.

15. Consider the quadratic equation

$p + qt + rt^2 + st^3 = t^4$

Or, $t^4 - st^3 - rt^2 - qt - p = 0.$

Now, by our assumption of the problem, a_1, a_2, a_3 and a_4 are the solution of this equation and hence

$s_1 = a_1 + a_2 + a_3 + a_4 = s$

$s_2 = (a_1 + a_2)(a_3 + a_4) + a_1a_2 + a_3a_4 = -r$

$s_3 = a_1a_2(a_3 + a_4) + a_3a_4(a_1 + a_2) = q$

$s_4 = a_1a_2a_3a_4 = -p$

The second system of equation is

$(t^2)^4 - w(t^2)^3 - z(t^2)^2 - y(t^2) - x = 0$

Putting $t^2 = u$, we have

$u^4 - wu^3 - zu^2 - yu - x = 0$

And the roots can be seen to be $a_1{}^2, a_2{}^2, a_3{}^2 \ and \ a_4{}^2$

And $s_1 = a_1{}^2 + a_2{}^2 + a_3{}^2 + a_4{}^2 = w$

$\Rightarrow w = (\sum a_i)^2 - 2\sum_{i<j} a_ia_j = s^2 + 2r$

$$s_2 = \sum_{i<j} a_i{}^2 a_j{}^2 = -z$$

$$or, z = -\sum_{i<j} a_i{}^2 a_j{}^2$$

$$= -(\sum_{i<j} a_ia_j)^2 + 2(\sum a_i)$$

$$\sum_{i<j<k} a_ia_ja_k - 2a_1a_2a_3a_4$$

For $(a_1{}^2a_2{}^2 + a_1{}^2a_3{}^2 + a_1{}^2a_4{}^2 + a_2{}^2a_3{}^2 + a_2{}^2a_4{}^2 + a_3{}^2a_4{}^2) = (a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4)^2 - 2(a_1 + a_2 + a_3 + a_4)(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4) - 2a_1a_2a_3a_4$

Hence z $= -r^2 + 2qs + 2p.$

$s_3 = a_1{}^2a_2{}^2a_3{}^2 + a_1{}^2a_2{}^2a_4{}^2 + a_1{}^2a_3{}^2a_4{}^2 + a_2{}^2a_3{}^2a_4{}^2 = y$

$y = (a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4)^2 - 2(a_1a_2a_3a_4)(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4)$

$= q^2 - 2pr$

Finally, $s_4 = a_1{}^2a_2{}^2a_3{}^2a_4{}^2 = -x$

$or, x = -(a_1{}^2a_2{}^2a_3{}^2a_4{}^2) = -(a_1a_2a_3a_4)^2 = -p^2$

$\therefore x = -p^2, y = q^2 - 2pr, z = -r^2 + 2qs + 2p$

And $w = s^2 + 2r$ is the solution.

16. $2^m = (1+1)^m = C_0 + C_2 + \cdots + C_m \ for \ m$ =1, 2, …n + 1

Now consider the polynomial

$$f(x) = 2\left[\binom{x-1}{0} + \binom{x-1}{1} + \binom{x-1}{2} + \cdots + \binom{x-1}{n}\right]$$

$$where\ \binom{x-1}{r} = \frac{(x-1)(x-2)\ldots(x-r)}{r}$$

Clearly, f(x) is of degree n.

Now

$$f(r) = 2\left[\binom{r-1}{0} + \binom{r-1}{1} + \cdots + \binom{r-1}{r-1} + \binom{r-1}{r} + \cdots + \binom{r-1}{n}\right] where\ 1 \le r \le n+1.$$

But $\binom{r-1}{k} = 0$ for all k> r−1 where k and are integers.

$= 2.2^{r-1} = 2^r\ for\ all\ r = 1, 2, \ldots, n+1.$

∴ Thus, f(x) is the required polynomial.

$$\therefore f(n+2) = 2\left[\binom{n+1}{0} + \binom{n+1}{1} + \cdots + \binom{n+1}{n} + \binom{n+1}{n+1} - 1\right]$$

$= 2[2^{n+1} - 1] = 2^{n+2} - 2$

Similarly, P(x+2) = $2^{x+2} - 2$.

17. If $k = 1; 1 < x < 2$ ....(1)

$k = 2; 2 < x^2 < 3$ ....(2)

$k = 3; 3 < x^3 < 4$ ....(3)

$k = 4; 4 < x^4 < 5$ ....(4)

$k = 5; 5 > x^5 > 6$ ....(5)

....

Consider the inequality $2 < x^2 < 3$, then x should lie between $\sqrt{2}$ and $\sqrt{3}$.

i.e. $\sqrt{2} < x < \sqrt{3}$.

Now, $1 < \sqrt{2} < x < \sqrt{3} < \sqrt{4} = 2$

And hence satisfies eq. (1) and eq. (2) of the inequalities

$$\sqrt{2} < x < \sqrt{3}$$

$$\Rightarrow \left(\sqrt{2}\right)^3 < x^3 < \left(\sqrt{3}\right)^3$$

$$\Rightarrow 2\sqrt{2} < x^3 < 3\sqrt{3}$$

Applying this in eq. (3),

$$3 < 2\sqrt{2} < x^3 < 3\sqrt{3}$$

$$But\ 3\sqrt{3} > 4$$

Now picking up the third inequality, we have

$$3 < x^3 < 4$$

$$\Rightarrow \sqrt[3]{3} < x\sqrt[3]{4}$$

$$\Rightarrow \sqrt[3]{9} < x^2 < \sqrt[3]{16} = 2\sqrt[3]{2}$$

Since $2^3 < 9, 2 < \sqrt[3]{9} < x^2 < \sqrt[3]{16} \nless \sqrt[3]{16} = 3$.

Again this value of x falls as x may be lying between $3 = \sqrt[3]{27}$ and $\sqrt[3]{16}$. Now we have the same problem with lower limit of x^4 for the 4th inequality.

So, trying with $\sqrt[3]{3} < x < \sqrt[4]{5}$, we shall get $3^4<x^{12}<5^3$ or $81<x^{12}<125$.

$1 < \sqrt[3]{3} < x < \sqrt[4]{5} < \sqrt[4]{16} = 2$ is true.

$2 = \sqrt[3]{8} < \sqrt[3]{9} < x^2 < \sqrt[4]{5^2} < \sqrt[4]{3^4} = 3$

Hence the second of the inequalities also true.

Again by our choice of x,

$\sqrt[3]{3} < x$

$\Rightarrow 3 < x^3 < \sqrt[4]{5^3} (= \sqrt[4]{125}) < \sqrt[4]{4^4} = 4$ ($\because$ $125 < 4^4 = 256$)

For the 4th row,

$\sqrt[3]{4^3} < \sqrt[3]{3^4} < x^4 < \sqrt[4]{5^4} = 5$ is again true.

For the 5th row,

$\sqrt[3]{3^5} < x^5 < \sqrt[4]{5^5}$

$\sqrt[3]{5^3} < \sqrt[3]{3^5} < x^5 < \sqrt[4]{5^5} \nleq \sqrt[4]{6^4}$

($\because \sqrt[4]{5^5} = \sqrt[4]{3125} \nleq \sqrt[4]{1296}$)

However, for the third row,

We have $3 < x^3$ and for the fifth row and we have, $x^5 < 6$

So $3^5 < x^{15} < 6^3$ for such x.

But this leads to the contradiction $243 < x^{15} < 216$ and hence the greatest n for which the rows of the given inequalities hold is 4 and for any x such that $\sqrt[3]{3} < x < \sqrt[5]{5}$ will satisfy these inequalities.

18. We have $f: N \to Z$.

And f(m) > f(n) for m > n.

$\Rightarrow f(1) < f(2) = 2$

$2 = f(2) = f(1 \times 2) = f(1) \times f(2) = f(1) \times 2$

$\therefore f(1) = \frac{2}{2} = 1$

Now, $f(4) > f(3) > f(2) = 2$

And $f(4) = f(2) \times f(2) = 2 \times 2 = 4$

And so 4 > f(3) > 2 and f(3) is an integer and hence f(3) = 3

And f(6) > f(5) > f(4)

$\Rightarrow f(2) \times f(3) > f(5) > f(2) \times f(2)$

$\Rightarrow 2 \times 3 > f(5) > 4$

$\Rightarrow f(5) = 5$

So we guess that f(n)= n.

Let us prove it.

We will use mathematical induction for proving.

f(n) = n is true for $n = 1,2$.

Let us assume that the result is true for all m<n and then we shall prove it for n, where n > 2.

If n is even then let n = 2m

$f(n) = f(2m) = f(2) \times f(m) = 2 \times m = 2m = n$

If n is odd and n=2m+1, then n > 2m.

$f(n) = f(2m+1) > f(2m) = 2m$

And $f(2m+2) = f2(m+1) = f(2) \times f(m+1)$

$n = 2m + 1$

m+1 < n, m < n

and hence f(2m+2)= $f[2(m+1)] = f(2) \times f(m+1) = 2 \times (m+1)$

Thus, 2m< f(2m+1) < f(2m+2) = $2(m+1) = 2m$+2

$\Rightarrow 2m < f(2m+1) < 2m+2$

There is exactly one integer 2m+1 between 2m and 2m+2 and hence

$f(n) = f(2m+1) = (2m+1) = n$

Thus $f(n) = n for\ all\ n \in N$

Hence f(1998) = 1998.

19. Let the number of pages in the novel be n. Since, the number of pages after a leaf is firm is 15,000, the sum of the numbers on all the pages must exceed 15,000.

$i.e., \frac{n(n+1)}{2} > 15,000$

$\Rightarrow n(n+1) > 30,000$

$\therefore (n+1)^2 > n(n+1) > 30,000 > 173^2$

$\Rightarrow (n+1) > 173$

$\Rightarrow n > 172$ ....(1)

The sum of the numbers on the page torn should be less than or equal to (n−1)+n= $2n - 1$.

Hence (1 + 2 + … + n)−(2n−1)≤ 15,000

$\Rightarrow n(n+1) - 2(2n-1) \leq 30,000$

$\Rightarrow n^2 - 3n + 2 \leq 30,000$

$\Rightarrow (n - 2(n-1) \leq 30,000$

$\Rightarrow (n-2)^2 < (n-2)(n-1) \leq 30,000 < 174^2$

$\Rightarrow (n-2) < 174$

$\Rightarrow n < 176$ ....(2)

By eq. (1) and (2), we get

172<n<176

So n could be one of 173, 174 or 175.

If n = 173, then

$$\frac{n(n+2)}{2} = \frac{173 \times 174}{2} = 15{,}051$$

Thus the sum of the numbers on the torn pages $= 15015 - 15{,}000$ =51 and this could be x + (x+1)= $2x + 1 = 51$.

So, the page numbers on the torn pages = $\frac{51+1}{2} = 26\ and \frac{51-1}{2} = 25$

If $n = 174$, then

$$\frac{n(n+1)}{2} = \frac{174 \times 175}{2} = 15{,}225.$$

So the sum of the numbers on the torn pages $15225 - 15000 = 225$ and in this case, the numbers on the torn pages = $\frac{225-1}{2} = 112\ and \frac{225+1}{2} = 113.$

(But actually the smaller number on the torn page should be odd and hence, though it is theoretically correct, but no acceptable in reality).

If $n = 175, then \frac{n(n+1)}{2} = \frac{175 \times 176}{2} = 15{,}400$ and the sum of the numbers on the torn page is 400 $(= 15400 - 15000)$ which is not possible because the sum should be an odd number and hence this value of n also should be rejected.

So the numbers on the torn page should be 25 and 26 and the number of pages is 173.

20. Case 1: n =4k + 1

$$n = 4k + 1 = (4k+1) \times (1)^{2k} \times (-1)^{2k}$$

$$= (4k+1) + \underbrace{(1 + 1 + \cdots + 1)}_{2k\ times} + \underbrace{[(-1) + (-1) + \cdots + (-1)}_{2k\ times}$$

Case 2: $n = 4l$ here there are two cases where (a) l is even with $l \geq 2$ and (b) l is odd with $l \geq 3$

(a) $n = 4l, l$ is even.

Consider integers w and v such that,

$$n = 4l = 2l \times 2 \times (1)^w \times (-1)^v$$
$$= 2l + 2 + \underbrace{(1 + 1 + \cdots + 1)}_{w\ times}$$
$$+ \underbrace{[(-1) + (-1) + \cdots + (-1)]}_{v\ times}$$

Now by definition of good integer, we have 2 + w + v= $4l$
[there are 2 + w + v factors).
$\Rightarrow w + v = 4l - 2$ ……(1)

And again since $4l = 2l + 2 + w - v$
We get $w - v = 2l - 2$ ……….(2)
Solving eq. (1) and eq. (2), we get $w = 3l - 2$ and $v = l$.

(b) l is odd. With $l \geq 3$.

Choose w and v such that

$$n = 4l = (2l) \times (-2) \times (1)^w(-1)^v$$

$$= 2l + (-2) + \underbrace{(1 + 1 + \cdots + 1)}_{w\ times}$$
$$+ \underbrace{[(-1) + (-1) + \cdots + (-1)]}_{v\ times}$$

Again since there are w + v + 2 factors,

We have w + v + 2 = $4l$

Or $w + v = 4l - 2$

And $4l = 2l - 2 + w - v$ (by definition of good integer)

$\Rightarrow w - v = 2l + 2$

Solving $w = 3l$ and $v = l - 2$

Since l is odd and $l \geq 3$

$l - 2 \geq 1$

Now, n = $4l = 2l \times (-2) \times (1)^{3l} \times (-1)^{l-2} = 2l + (-2) + \underbrace{(1 + 1 + \cdots + 1)}_{3l\ times} + \underbrace{[(-1) + (-1) + \cdots + (-1)}_{(l-2) times}$

$= 2l - 2 + 3l - (l - 2) = 4l$

21. Without loss of generality, we may assume $a \geq b \geq c$, so that $|c^2 - a^2| = a^2 - c^2$ is the maximum of $|a^2 - b^2|, |b^2 - c^2| and\ |c^2 - a^2|$.

It is enough to prove that $a^2 + b^2 + c^2 - \sqrt{3}(a^2 - b^2) > 0$

Now $a^2 + b^2 + c^2 - \sqrt{3}(a^2 - c^2) > a^2 + (a - c)^2 + c^2 - \sqrt{3}(a^2 - c^2)$

(as b > a –c, by triangle inequality)

$$> 2a^2 + 2c^2 - 2ac - \sqrt{3}a^2 + \sqrt{3}c^2$$
$$> (2 - \sqrt{3})a^2 + (2 + \sqrt{3})c^2 - 2ac$$

But $(\sqrt{3} - 1)^2 = 2(2 - \sqrt{3})$

And $(\sqrt{3} + 1)^2 = 2(2 + \sqrt{3})$

So $a^2 + b^2 + c^2 - \sqrt{3}(a^2 - c^2) > \frac{[(\sqrt{3}-1)a]^2 - 4ac + [(\sqrt{3}+1)c]^2}{2} = \frac{1}{2}\left[(\sqrt{3} - 1)a - (\sqrt{3} + 1)c\right]^2 \geq 0$

And hence the result.

22. By checking the first four values we find $3^{1/3}$ to be the largest. We will prove that $\left\{n^{\frac{1}{n}}\right\}, n \geq 3$ is a decreasing sequence.

$$n^{\frac{1}{n}} > (n+1)^{\frac{1}{n+1}}$$

$\Leftrightarrow n^{n+1} > (n+1)^n$

$\Leftrightarrow n > \left(1+\frac{1}{n}\right)^n$

Now, $\left(1+\frac{1}{n}\right)^n = 1 + n.\frac{1}{n} + \frac{n(n-1)}{2} \times \frac{1}{n^2} + \frac{n(n-1)(n-2)}{6}.\frac{1}{n^3} + \cdots$

$= 1 + 1 + \frac{1}{2}\left(1-\frac{1}{n}\right) + \frac{1}{6}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots$

$< 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots < 3$

$or, 3 > \left(1+\frac{1}{n}\right)^n$

$\therefore if\ n \geq 3, n^{\frac{1}{n}} > (n+1)^{\frac{1}{n+1}}$

i.e., $\{n^{\frac{1}{n}}\}$ is decreasing for $n \geq 3$.

But $3^{1/3}$ is also greater than 1 and $2^{1/2}$. Hence $3^{1/3}$ is the largest.

23. Taking $k = 1$, since

$a_1^k + a_2^k + \cdots + a_n^k \geq 0,$

And for $k = 1$, we have

$a_1 + a_2 + \cdots + a_n \geq 0$……..(1)

And since $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n, a_1 \geq 0$ ……..(2)

And if all $a_1, i = 1,2, \ldots n$ are positive a_1 is the maximum of all $a_i s$

$\therefore p = |a_1| = a_1$

Suppose that some of the $a_i s$ are negative and p≠ a_1, then $a_n < 0$ and hence

$p = |a_n|$

Let r be an index such that

$a_n = a_{n-1} = \cdots = a_{r+1} < a_r \leq a_{r-1} \leq \cdots \leq a_1$

Then $a_1^k + a_2^k + \cdots + {a_{r-1}}^k + a_r^k + \cdots + a_n^k$

$$= a_n^k\left\{\left(\frac{a_1}{a_n}\right)^k + \left(\frac{a_2}{a_n}\right)^k + \cdots + \left(\frac{a_{r-1}}{a_n}\right)^k + \left(\frac{a_r}{a_n}\right)^k + (n-r)\right\} = a_n^k X$$

Where the value of the second bracket is taken as X.

Since $\left|\frac{a_1}{a_n}\right|, \left|\frac{a_2}{a_n}\right|, \ldots, \left|\frac{a_r}{a_n}\right|$ are all less than 1, so their kth powers are all less than these fractions and by taking k sufficiently large, which would make X > 0 and $Xa_n^k < 0$ for k odd, a contradiction and hence $p = a_1$.

Taking $x > a_1$, then by AM−GM inequality,

$(x-a_2)(x-a_3)(x-a_4)\ldots(x-a_n)$

$$\leq \left(\frac{\sum_{j=2}^{n}(x-a_j)}{n-1}\right)^{n-1}$$

$$\leq \left(\frac{(n-1)x + a_1}{n-1}\right)^{n-1} \qquad [\because \sum_{i=1}^{b} a_i \geq 0]$$

$$= \left(x + \frac{a_1}{n-1}\right)^{n-1}$$

$$\leq x^{n-1} + x^{n-2}.a_1 + x^{n-2}a_1^2 + \cdots + a_1^{n-1}$$

[Here we have used $\binom{n-1}{r} \leq (n-1)^r, r \geq 1$]

Multiplying both sides by $(x-a_1)$, we get

$(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n) \leq (x-a_1)(x^{n-1} + x^{n-2}a_1 + \cdots + a_1^{n-1} = x^n - a_1^n$.

24. $a_0 = 1\ and\ a_1 = a > 2$ and so a can be written as

$$b + \frac{1}{b} = \frac{b^2 + 1}{b}$$

For real number b > 0 and

$$a^2 - 2 = b^2 + \frac{1}{b^2}$$

$$now, a_2 = \left(\frac{a_1^2}{a_0^2} - 2\right) a_1 = \left(\frac{a^2}{1} - 2\right) a$$
$$= (a^2 - 2)a$$

$$= \left(b^2 + \frac{1}{b^2}\right)\left(b + \frac{1}{b}\right) = \left(b^3 + \frac{1}{b^3} + b + \frac{1}{b}\right)$$

$$= \frac{b^6 + 1 + b^4 + b^2}{b^3} = \frac{(b^2+1)(b^4+1)}{b^3}$$

Similarly,

$$a_3 = \left[\left(\frac{a^2}{a^1}\right)^2 - 2\right] a_2 \;= \left[\left(b^2 + \frac{1}{b^2}\right)^2 - 2\right] a_2$$

$$= \left[\left(b^2 + \frac{1}{b^2}\right)^2 - 2\right]\left(b^2 + \frac{1}{b^2}\right)\left(b + \frac{1}{b}\right)$$

$$= \left(b^4 + \frac{1}{b^4}\right)\left(b^2 + \frac{1}{b^2}\right)\left(b + \frac{1}{b}\right)$$

$$\left(b^{2^2} + \frac{1}{b^{2^2}}\right)\left(b^{2^1} + \frac{1}{b^{2^1}}\right)\left(b^{2^0} + \frac{1}{b^{2^0}}\right)$$

$$(b^2+1)(b^4+1)(b^8+1)/b^7$$

And proceeding in this manner, we get

$$a_n = \left(b^{2^{n-1}} + \frac{1}{b^{2^{n-1}}}\right)\left(b^{2^n} + \frac{1}{b^{2^n}}\right) \dots \left(b + \frac{1}{b}\right)$$

Hence, L.H.S. $= \sum_{i=0}^{n} 1/a_i$

$$= 1 + \frac{b}{b^2+1} + \frac{b^3}{(b^2+1)(b^4+1)} + \frac{b^7}{(b^2+1)(b^4+1)(b^8+1)} + \cdots + \frac{b^{2^{n-1}}}{(b^2+1)(b^4+1)\dots(b^{2^n}+1)}$$

The right hand side of the inequality is

$$\frac{1}{2}\left(a + 2 - \sqrt{a^2 - 4}\right)$$
$$= \frac{1}{2}\left[b + \frac{1}{b} + 2 - \left(b - \frac{1}{b}\right)\right]$$
$$= \left(\frac{1}{b} + 1\right)$$

You may know the following identity:

$$\sum_{j=1}^{n} \frac{a_j}{(1+a_1)\dots(1+a_n)} = 1 - \frac{1}{(1+a_1)\dots(1+a_n)}$$

[This result is obtained by using partial fractions].

So the L.H.S.

$$\sum_{i=1}^{n} \frac{1}{a_i}$$
$$= 1 + \frac{b}{b^2+1} + \frac{b^3}{(b^2+1)(b^4+1)} + \cdots + \frac{b^{2^n - 1}}{(b^2+1)(b^4+1)\dots(b^{2^n}+1)}$$

$$\sum_{i=1}^{n} \frac{b^{2^i}}{(b^2+1)\dots(b^{2^i}+1)} = 1 - \frac{1}{(b^2+1)(b^4+1)\dots(b^{2^n}+1)} < 1$$

Now

$$L.H.S. = \frac{1}{b}\left[b + \frac{b^2}{b^2+1} + \frac{b^4}{(b^2+1)(b^4+1)} + \cdots + \frac{b^{2^n}}{(b^2+1)\dots(b^{2^n}+1)}\right]$$

$$= 1 + \frac{1}{b}\left[\frac{b^2}{b^2+1} + \cdots + \frac{b^{2^n}}{(b^2+1)\dots(b^{2^n}+1)}\right]$$

$$\textit{and clearly } \frac{b^2}{b^2+1}+\frac{b^4}{(b^2+1)(b^4+1)}+\cdots+\frac{b^{2^n}}{(b^2+1)(b^{2^n}+1)}$$

$$\sum_{i=1}^{n}\frac{b^{2^i}}{(1+b^2)\ldots(1+b^{2^i})}=1-\frac{1}{(1+b^2)\ldots(1+b^{2^n})}$$

So, the L.H.S.

$$=\sum_{i=0}^{n}\frac{1}{a_i}=1+\frac{1}{b}\left[\sum_{i=1}^{n}\frac{b^{2^i}}{(1+b^2)\ldots\left(1+b^{2^i}\right)}\right]=1+\frac{1}{b}\left(1-\frac{1}{(1+b^2)\ldots(1+b^{2^n})}\right)$$

$$=1+\frac{1}{b}-\frac{1}{b(1+b^2)(1+b^4)\ldots(1+b^{2^n})}<1+\frac{1}{b}=R.H.S.$$

And hence the result.

25. Considering the polynomial $\pm P(\pm x)$ we may assume without loss of generality that a, b ≥ 0.

Case I : if c, d ≥ 0, then

$P(1)=a+b+c+d\leq 1<7$

Case II: if $d\ \leq 0$ and $c\geq 0$, then |a|+|b|+|c|+|d|$=a+b+c+d=(a+b+c+d)-2d$

$=P(1)-2P(0)\leq 1+2=3<7$

Case III: If $d\geq 0, c<0$

|a|+|b|+|c|+|d| $=a+b-c+d$

$$=\frac{4}{3}P(1)-\frac{1}{3}P(-1)-\frac{8}{3}P\left(\frac{1}{2}\right)+\frac{8}{3}P\left(-\frac{1}{2}\right)$$

$$\leq\frac{4}{3}+\frac{1}{3}+\frac{8}{3}+\frac{8}{3}=\frac{21}{3}=7$$

Case IV: d > 0, c > 0

|a|+|b|+|c|+|d| $=a+b-c-d$

$$\frac{5}{3}P(1)-4P\left(\frac{1}{2}\right)+\frac{4}{3}P\left(-\frac{1}{2}\right)$$

$$\leq\frac{5}{3}+4+\frac{4}{3}=\frac{21}{3}=7$$

26. Here we use A.P. $\geq$ G.P.

$$a=\frac{(a+b-c)+(a-b+c)}{2}\geq\sqrt{(a+b-c)(a-b+c)}$$

$$b=\frac{(b+a-c)+(b-a+c)}{2}\geq\sqrt{(b+a-c)(b-a+c)}$$

$$c=\frac{(c+a-b)+(c-a+b)}{2}\geq\sqrt{(c+a-b)(c-a+b)}$$

$$\therefore a.b.c.=\frac{[(a+b-c)+(a-b+c)][(b+a-c)+(b-a+c)][(c+a-b)+(c-a+b)]}{8}$$

$$\geq\sqrt{(a+b-c)-(a-b+c)(b+a-c)(b-a+c)(c+a-b)(c-a+b)}$$

$$=(a+b-c)(b-a+c)(c-b+a)$$

i.e., $abc\geq(a+b-c)(b-a+c)(c-b+a)=(2-2c)(2-2a)(2-2b)=8(1-c)(1-a)(1-b)[\because a+b+c=2]$

$$\therefore\frac{a}{1-a}.\frac{b}{1-b}.\frac{c}{1-c}\geq 8$$

27. Taking $x = 1$ in the given equation :

$(1 + x + x^2)^{25} = a_0 + a_1x + a_2x^2 + \cdots + a_{50}x^{50}$

We get

$$3^{25} = a_0 + a_1 + a_2 + \cdots + a_{50}$$

Similarly, $x = -1$ gives

$$1 = a_0 + a_1 + a_2 + \cdots + a_{50}$$

Adding these, we have

$$1 + 3^{25} = 2(a_0 + a_2 + a_4 + \cdots + a_{50})$$

But

$$1 + 3^{25} = 3^{25} - 1 + 2$$

$$= 2(3^{24} + 3^{23} + 3^{22} + \cdots + 1 + 1)$$

There are even number of odd terms in the braces, and hence the sum is even. This implies that $a_0 + a_2 + a_4 + \cdots + a_{50}$ is even.

28. Assume, if possible

$f(x) = (x + a)(x^3 + ax^2 + bx + c)$

Comparing the coefficients of like powers of x, we get

$$a + b = 26$$

$$ab + c = 52,$$

$$ac + d = 78,$$

$$ad = 1989$$

But $1989 = 3^2.13.17$. Thus 13 divides ad and hence 13 divides a or d but not both. If 13 divides a then 13 divides $d = 78 - ac$ which is not possible. Suppose 13 divides d. Then 13 divides ac. But since 13 does not divide a, 13 divides c which implies 13 divides $ad = 52 - c$ and so b is divisible by 13 which in turn implies 13 divides $a = 26 - b$, a contradiction.

Therefore, f(x) has no linear factors:

If $f(x) = (x^2 + ax + b)(x^2 + cx + d)$, then again

$a = c = 26,$

$b + ac + d = 52,$

$ad + bc = 78$

$bd = 1989$

Since 1989= $3^2.13.17$, 13 divides bd. This implies that 13 divides b or d but not both. If 13 divides b, the 13 divides ad $(= 78 - bc)$ and hence 13 divides a. But then 13 divides d $(= 52 - b - ac)$ a contradiction. Similar argument shows that 13 divides d is also not possible. We conclude that f(x) cannot be written as a product of two polynomials with integral coefficients, each of degree < 4.

29. Consider 2001 numbers

$$\frac{1}{k}, 1001 \leq k \leq 3001$$

Using AM−HM inequality, we get

$$\left(\sum_{k=1001}^{3001} k\right)\left(\sum_{k=1001}^{3001} \frac{1}{k}\right) > (2001)^2$$

But

$$\sum_{k=1001}^{3001} k = (2001)^2$$

Hence we get the inequality.

$$\sum_{k=1001}^{3001} \frac{1}{k} > 1$$

On the other hand, grouping 500 terms at a time, we also have

$$S = \sum_{k=1001}^{3001} \frac{1}{k}$$

$$< \frac{500}{1000} + \frac{500}{1500} + \frac{500}{2000} + \frac{500}{2500} + \frac{1}{3001}$$

$$< \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{3000} = \frac{3851}{3000} > \frac{4}{3}$$

Note : We can sharpen the above inequality. Consider the sum

$$S = \sum_{k=n+1}^{3n+1} \frac{1}{k}$$

There are 2n+1 terms in the sum and the middle term is $\frac{1}{2n+1}$. We can write the sum in the form

$$S = \frac{1}{2n+1} + \sum_{k=1}^{n} \left(\frac{1}{2n+1+k} + \frac{1}{2n+1-k} \right)$$

$$= \frac{1}{2n+1} + \frac{2}{(2n+1)} \sum_{k=1}^{n} \frac{1}{1 - \left(\frac{k}{2n+1} \right)^2}$$

For 0 < a < ½, we have

$$1 + a < \frac{1}{1-a} < 1 + 2a$$

Thus we get the bounds

$$\frac{1}{2n+1} + \frac{2}{2n+1} \sum_{k=1}^{n} \left[1 + \left(\frac{k}{2n+1} \right)^2 \right] < S$$

And

$$S < \frac{1}{2n+1} + \frac{2}{2n+1} \sum_{k=1}^{n} \left[1 + 2 \left(\frac{k}{2n+1} \right)^2 \right].$$

This on simplification gives

$$1 + \frac{2}{(2n+1)^3} \sum_{k=1}^{n} k^2 < S$$

$$< 1 + \frac{4}{(2n+1)^3} \sum_{k=1}^{n} k^2.$$

Now using the identity

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

The inequality simplifies to

$$1 + \frac{n(n+1)}{3(2n+1)^2} < S < 1 + \frac{2n(n+1)}{3(2n+1)^2}$$

But for $n \geq 1$, we also have

$$\frac{2}{9} \leq \frac{n(n+1)}{(2n+1)^2} \leq \frac{1}{4}$$

This leads to

$$\frac{29}{27} < S < \frac{7}{6}$$

30. We have,

$$y + z = 4 - x$$

$$y^2 + z^2 = 6 - x^2$$

From Cauchy Schwarz inequality we get,

$y^2 + z^2 \geq \frac{1}{2}(y+z)^2$

Hence,

$$6 - x^2 \geq \frac{1}{2}(4 - x)^2$$

This simplifies to (3x−2)(x−2)≤ 0. Hence we have 2/3 $\leq x \leq 2$.

Suppose $x = 2$.

Then $y + z = 2, y^2 + z^2 = 2$ which has solution $y = z = 1$. (Similarly $x = 2/3$ is also possible (verify)).

Since the given relations are symmetric in x, y and z, similar assertions hold for y and z also.

31. Consider the polynomial f(x) −2. This vanishes at a_1, a_2, a_3, a_4 and a_5. Hence

$$f(x) - 2 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)(x - a_5)g(x)$$

For some polynomial g(x) with integral coefficients.

If f(b) = 9 for some integer b, then

$7 = (b - a_1)(b - a_2)(b - a_3)(b - a_4)(b - a_5)g(b)$

which is impossible because the integers $b - a_1, b - a_2, \ldots, b - a_5$ are all distinct and 7 cannot be factored into more than 3 distinct numbers. [Best we can do is $7 = (-7)(-1)(1)$].

Remark : The same conclusion holds even if f(x) assumes the value 2 for only 4 distinct integers.

32. The relation (i) gives

$$f(1990) - 90\left[\frac{f(1990)}{90}\right] = 1990 - 19\left[\frac{1990}{19}\right] = 1990 - 1976 = 14$$

Using relation (ii),

$$\frac{1990}{90} < \frac{f(1990)}{90} < \frac{2000}{90}$$

$$or, 21\frac{10}{90} < \frac{f(1990)}{90} < 22\frac{20}{90}$$

Thus

$$\left[\frac{f(1990)}{90}\right] = 21 \text{ or } 22$$

If

$$\left[\frac{f(1990)}{90}\right] = 21$$

Then

$f(1990) = 14 + 90.21 = 1904$

If

$$\left[\frac{1990}{90}\right] = 22$$

Then

$f(1990) = 1994$

33. Since the right hand side is positive, so is the left hand side. Hence x must be positive.

Let $x = n + f$, where $n = [x]$ and $f = [x]$. We consider two cases:

Case 1: $0 \leq f < 1/2$: In this case, we get [2x]= $[2n + 2f] = 2n$, as 2f < 1. Hence the equation becomes

$$\frac{1}{n} + \frac{1}{2n} = f + \frac{1}{3}$$

This forces (1/n)+(1/2n) ≥ 1/3. We conclude that $2n - 9 \leq 0$. Thus n can take values 1, 2, 3, 4. Among these $n = 2{,}3{,}4$ are all admissible, because for n = 2,3,4 we get f=5/12, 1/6, 1/24 respectively which are all less than ½; while $n =$

1 is not admissible, because $n = 1$ gives f > ½. We get three solutions in this case;

$$x = 2 + \left(\frac{5}{12}\right) = \frac{29}{12}; x = 3 + \left(\frac{1}{6}\right) = \frac{19}{6}; x = 4 + \left(\frac{1}{24}\right) = \frac{97}{24}$$

Case 2: (1/2) ≤ f<1: Now we get [2x] $= 2n + 1$, as 1≤ $2f < 2$. The given equation reduces to

$$\frac{1}{n} + \frac{1}{2n+1} = f + \frac{1}{3}$$

We conclude, as in Case 1. 1/n + 1/ (2n+1) ≥ 1/2 + 1/3. This reduces to $10n^2 - 13n - 6 \leq 0$. It follows that $n = 1$. But this is not admissible since n = 1 gives $f = 1$. We do not have any solution in this case.

34. By looking at the first few values of a_n, we guess that

$a_n = (n-1)^2 + 1 = n^2 - 2n + 2$.

We prove this by induction on n. In fact,

$$a_{n+1} = 2a_n - a_{n-1} + 2$$

$$= 2[(n-1)^2 + 1] - [(n-2)^2 + 1] + 2$$

$$= 2n^2 - 4n + 4 - (n^2 - 4n + 5) + 2 = n^2 + 1$$

Now we have,

$$a_m a_{m+1} = [(m-1)^2 + 1][m^2 + 1]$$

$$= m^2(m-1)^2 + m^2 + (m-1)^2 + 1$$

$$= [m(m-1) + 1]^2 + 1$$

$$= a_{m^2 - m + 2}$$

35. Let α, β, γ be the rots of the given cubic $x^3 - ax + b = 0$, where a > 0 and b > 0. We have

$$\left.\begin{aligned} \alpha + \beta + \gamma &= 0 \\ \alpha\beta + \beta\gamma + \gamma\alpha &= -a \\ \alpha\beta\gamma &= -b \end{aligned}\right\}$$

From the last of these equations, we see that either all the roots are negative or two are positive and one negative. However, the second equation in (*) shows that all three cannot be negative. So, two of α, β, γ are positive and the remaining root is negative. The first equation in (*) implies that the negative rot is numerically larger than the other two positive roots. Hence, we may assume that $\gamma < 0 < \alpha \leq \beta \text{ where } |\alpha| \leq |\beta| \leq |\gamma|$.

We have

$b - a\alpha = -\alpha\beta\gamma + \alpha(\alpha\beta + \beta\gamma + \gamma\alpha) = \alpha^2(\beta + \gamma) = -\alpha^3 < 0.$

Since a is positive, we get b/a < α proving the first inequality.

Again, we have

$3b - 2a\alpha = -3\alpha\beta\gamma + 2\alpha(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= -\alpha\beta\gamma + 2\alpha^2\beta + 2\alpha^2\gamma$

$\alpha[2\alpha(\beta + \gamma) - \beta\gamma]$

$= \alpha[-2(\beta + \gamma)^2 - \beta\gamma] \qquad (since\ \alpha = -(\beta + \gamma)$

$= -\alpha(2\beta^2 + 5\beta\gamma + 2\gamma^2)$

$= -\alpha(2\beta + \gamma)(\beta + 2\gamma)$

$= -\alpha(\beta - \alpha)(\gamma - \alpha).$

Observe that $-\alpha < 0, \beta \geq \alpha, \gamma - \alpha < 0$. Hence $3b - 2a\alpha$ is nonnegative. This proves the second inequality, $\alpha \leq 3b/2a$.

36. Since l is a root of the equation $x^3 + ax^2 + bx + c = 0$,

We have

$l^3 = -al^2 - bl - c$

This implies that

$l^4 = -al^3 - bl^2 - cl = (1-a)l^3 + (a-b)l^2 + (b-c)l + c$

Where we have used again

$-l^3 - al^2 - bl - c = 0$.

Suppose $|l| \geq 1$. Then we obtain

$|l|^4 \leq (1-a)|l|^3 + (a-b)|l|^2 + (b-c)|l| + c$

$\leq (1-a)|l|^3 + (a-b)|l|^3 + (b-c)|l|^3 + c|l|^3$

$\leq |l|^3$.

This shows that $|l| \leq 1$. Hence the only possibility in this case is $|l| = 1$. We conclude that $|l| \leq 1$ is always true.

37. Let us take square of an even integer, say, 2a.

$N = 2a$

$\Rightarrow N^2 = 2a \times 2a = 4a^2$

And $4a^2$ is not in the form of 4n + 3 or 4n + 2.

If N is an odd number, then $N = 2a + 1$.

And $N^2 = (2a+1)^2 = 4a^2 + 4a + 1 = 4a(a+1) + 1 = 4n + 1$.

Here again the square is not in the form of 4n + 3 or 4n + 2. In other words, any number in the form of 4n + 3 or 4n + 2 cannot be a square number.

Where N is odd, $N^2 = 4a(a+1) + 1$.

As either a or a+1 is even, $N^2 = 8k + 1$ for some k ∈ N.

∴ The square of an odd number is in the form of 8k+1.

38. Let $n = 2^{m-1} \times p$, where $p = 2^m - 1$ is a prime number.

The divisors of $2^{m-1} \times p$ are 1, 2, 2^2, 2^3, $2^{m-1}, p, 2p, 2^2p, \ldots, 2^{m-2}p, 2^{m-1}p$.

Now, we should sum all these divisors excepting the last one, viz. $2^{m-1}p$.

The number of ways, in which a composite number can be expressed as a product of two factors, which are relatively prime to each other, is 2^{n-1}, where n is the number of distinct prime.

For example, $5^8 \times 3^7 \times 41^5$ can be resolved into product of two factors, in $2^{3-1} = 2^2 = 4$ ways so that factors are co−prime numbers.

Here they are

$$5^8 \times (3^7 \times 41^5)$$

$$3^7 \times (5^8 \times 41^5)$$

$$41^5 \times (3^7 \times 5^8)$$

And finally, $1 \times (41^5 \times 3^7 \times 5^8)$.

$$S = (1 + 2 + 2^2 + \cdots + 2^{m-1} + p(1 + 2 + 2^2 + \cdots + 2^{m-2})$$

$$= \frac{1(2^m - 1)}{2 - 1} + \frac{p[1(2^m - 1)]}{2 - 1}$$

$$= 2^m - 1 + p(2^{m-1} - 1)$$

$$2^m + p2^{m-1} - p - 1$$

$$= 2^{m-1}(2 + p) - (p + 1)$$

$$= 2^{m-1}(1 + 2^m) - 2^m \qquad [\because p = 2^m - 1]$$

$$= 2^{m-1}(2^m - 1) = n$$

Now s(n), the sum of positive divisors of n, is given by

$$s(n) = \frac{p_1^{a_1+1} - 1}{p_1 - 1} \times \frac{p_2^{a_2+1} - 1}{p_2 - 1} \times \ldots \times \frac{p_m^{a_m+1} - 1}{p_m - 1}$$

Where $n = p_1^{a_1} \times p_2^{a_2} \times p_m^{a_m}$

For example,

$$s(48) = s(2^4 \times 3)$$

$$= \frac{2^5 - 1}{2 - 1} \times \frac{3^2 - 1}{2 - 1} = 31 \times 4 = 124$$

$s_k(n)$, the sum of the kth power of the divisors of n = $\frac{p_1^{k(a_1+1)}-1}{p_1^k-1} \times \frac{p_2^{k(a_2+1)}-1}{p_2^k-1} \times \ldots \frac{p_m^{k(a_m+1)}}{p_m^k-1}$.

39. Let the divisor be d and the remainder be r.

Then by Euclidean Algorithm, we find

$19779 = dq_1 + r$ ……(1)

$17997 = dq_2 + r$ ………(2)

By subtracting eq. (2) from eq. (1), we get

$1782 = d(q_1 - q_2)$

∴ d is a three digit divisor of 1782.

Therefore, possible values of d are 891, 594, 297, 198.

Hence, largest three digit divisor is 891 and the remainder is 177.

40. (i) If $60 = 3 \times 4 \times 5$ and 4|100n, then 4 should divide $n^3 + 30n^2$ i.e., 4 should divide $n^2(n + 30)$. This implies that n is even.

(ii) If 5 $|(30n^2 + 100n)$, then 5 should divide n^3. Hence 5 should divide n.

(iii) If 3|30 n^2, then 3 should divide n^3+100n, i.e., 3 should divide $n(n^2 + 100n) = n(n^2 + 1 + 99)$

If $n \equiv \pm 1(mod\ 3), n^2 = 1$(mod 3) and

$n^2 + 1 \equiv 2(mod\ 3)$, so neither of $(n^2 + 1 + 99)$ and n are divisible by 3.

However, if $n = 0\ (mod\ 3), then\ n(n^2 + 1 + 99)$ is divided by 3, i.e., n(n^2+100) is divisible by 3 only if n is a multiple of 3. From (i) , (ii) and (iii), we find that n must be a multiple of 30. So, we should find the sum of all multiples of 30 less than 1998.

$S_n = 30 + 60 + \cdots + 1980 = 30(1 + 2 + \cdots + 66) = 66330$

Principles of Induction

1. First Principle of Mathematical Induction : Let {T(n): n∈ N} be a set of statements.

If T(1) is true and the truth of T(k) implies the truth of T(k+1), then T(n) is true for all n.

$$Example: S_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$S_1 = \frac{1 \times 2}{2} = 1 \; is\ true.$$

$$Let\ S_k = \sum_{t=1}^{k} t = \frac{k(k+1)}{2}$$

$$S_{k+1} = \sum_{t=1}^{k+1} t = \sum_{t=1}^{k} t + k + 1$$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{(k+1)(k+2)}{2}$$

Hence, the identity is true for all n by induction.

2. Second Principle of Mathematical Induction Strong principle of mathematical induction: Let {T(n) : $n \in N$} be a set of statements. If (i) T (1) is true and (ii) if for each natural number k, the truth of T(m) for all m < k where $k \geq 2$ implies the truth of T(k), then T(n) is true for all n.
3. Third Principle of Mathematical Induction. Let {T(n) |$n \in N$} be a set of statements for each natural number n. If

   (i) T (a) is true for some a ∈ N
   (ii) T(k) is true implies that T(k+1) is true for all $k \geq a$, then T(n) is true for all natural number $n \geq a$.

Examples on Mathematical Inductions

(1) There must be something wrong with the following proof: what is it?

Theorem : Let a be a positive number. For all positive integers n, we have $a^{n-1} = 1$.

Proof : If $n = 1, a^{n-1} = a^{1-1} = a^0 = 1$.

Assume that this statement is true for $n \leq k, i.e., a^{n-1} = 1$ for all $n \leq k$.

If $k \geq 1$ now for n $= k + 1$, we have

$$a^{(k+1)-1} = a^k = \frac{a^{k-1} \times a^{k-1}}{a^{k-2}} = \frac{1 \times 1}{1} = 1$$

So the theorem is true for $n = k + 1$ wherever the theorem is true for $n \leq k$ and hence, by the second principle of Mathematical induction, the theorem is true for all natural numbers, n.

Fallacy, for this explanation :

When we have written $a^{(k+1)-1}\ as\ \frac{a^{k-1} \times a^{k-1}}{a^{k-2}}$, we have assumed that the theorem is true for $n \leq k$ and we have verified that it is true for $n = 1$. For example, taking $k = 1$; the denominator becomes $a^{1-2} = a^{-1}$ but we have not proved that $a^{-1} = 1$; neither can it proved.

Therefore the proof has a loop hole here.

41. From the pattern

$1^3 = 1, 2^3 = 3 + 5, 3^3 = 7 + 9 + 11, 4^3 = 13 + 15 + 17 + 19$ …

Note that the first term on the R.H.S are 1st, 2nd, 4th, 7th, …, odd numbers. So the R.H.S. of the nth identity to be proved has

$\left[\frac{(n-1)n}{2} + 1\right] st$ odd number as first term.

Hence the nth identity to be proved is

$$n^3 = \left[\frac{(n-1)n}{2} + 1\right] 2 - 1 + \cdots n \; odd \; terms.$$

i.e. $n^3 = (n^2 - n + 1) + (n^2 - n + 3) + \cdots + (n^2 + n - 1)$(n terms) . Assume this is true for n

then (n+1)th identity to be proved is

$(n+1)^3 = (n^2 + n + 1) + (n^2 + n + 3) + \cdots + (n^2 + n + 2n + 1)$ [(n+1 terms)]

$= (n^2 - n + 1) + (n^2 - n + 3) + \cdots + (n^2 + n - 1) + 2n + \cdots + (n^2 + n + 2n + 1)$

$= n^3 + 2n^2 + n^2 + 3n + 1 = (n+1)^3$

So the right hand side of kth row is

$(k^2 - k + 1) + (k^2 - k + 3) + \cdots + [(k^2 - k + 1) + (k-1)2]$

$= (k^2 - k + 1) + (k^2 + k + 3) + \cdots + (k^2 + k - 1)$

Now if we assume that the pattern holds for the kth row then we will have

$(k^2 - k + 1) + (k^2 - k + 3) + \cdots + (k^2 + k - 1) = k^3$

Now the (k+1)th row numbers will be

$(k^2 + k + 1) + (k^2 + k + 3) + \cdots + (k^2 + k + 1 + 2k)$

$= (k^2 + k + 1) + (k^2 + k + 3) + \cdots + (k^2 + 3k + 1)$

$= [(k^2 - k + 1) + 2k] + [(k^2 - k + 3) + 2k] + \cdots + [(k^2 + k - 1) + 2k] + (k^2 + 3k + 1)$

$\underbrace{(k^2 - k + 1) + (k^2 - k + 3) + \cdots + (k^2 + k - 1)}_{k^3} +$
$(k \times 2k) + (k^2 + 3k + 1) + 3k^2 + 3k + 1 = (k+1)^3$ (By assumption)

Note : Now adding both the sides of n rows, we get

$1^3 + 2^3 + 3^3 + \cdots + n^3 = 1 + 3 + 5 + \cdots + (2n-1) + \cdots + (n^2 + n - 1)$

Thus, on the right side there are

$$\frac{(n^2 + n - 1) + 1}{2} = \frac{n(n+1)}{2} \; odd \; numbers \; from \; 1$$

$$so \; 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

[Sum of the first n odd numbers $= n^2$: Prove by Mathematical Induction

Hint: $1 + 3 + 5 + \cdots + 2n - 1 = n^2, n = 1 = 1 \times 1 = 1^2$ ...]

42. For $n = 10$, we have $2^{10} = 1024 > 10^3 = 1000$.

So the statement is true for $n = 10$.

Supposing that this statement is true for $n = k \geq 10$;

i.e., $2^k > k^3$.

For $n = k + 1$, $2^{k+1} > 2 \times k^3$

Now, $2k^3 - (k^3 + 3k^2 + 3k + 1) = k^3 - 3k^2 - 3k - 1 = (k-1)^3 - 6k.$

Let $k = 10 + a, where \; a \geq 0.$

Then $(k-1)^3 - 6k = (10 + a - 1)^3 - 6(10 + a) = (9 + a)^3 - 60 - 6a$

$= 729 + 243a + 27a^2 + a^3 - 60 - 60a$

$= 669 + 183a + 27a^2 + a^3 \geq 0$

$[\because a \geq 0]$

$\Rightarrow 2k^3 > (k+1)^3$

$\Rightarrow 2^{k+1} > (k+1)^3$

Hence the inequality is true for all $n \geq 10$.

43. Let us assume that the result is true for t_k for all k > n.

$$t_1 = \frac{1}{2}\left[\left(1+\sqrt{3}\right)^1 + \left(1-\sqrt{3}\right)^1\right]$$

$$= \frac{1}{2}\left(1+\sqrt{3}+1-\sqrt{3}\right)$$

$$= \frac{1}{2} \times 2 = 1 \textit{ is true}.$$

$$t_2 = 4 = \frac{1}{2}\left[\left(1+\sqrt{3}\right)^2 + \left(1-\sqrt{3}\right)^2\right] = \frac{1}{2}(8)$$
$$= 4 \textit{ is also true}.$$

Now, we have to prove that

$$t_n = \frac{1}{2}\left[\left(1+\sqrt{3}\right)^n + \left(1-\sqrt{3}\right)^n\right]$$

Since

$$t_n = 2[t_{n-1} + t_{n-2}]$$

$$= 2\left[\frac{1}{2}\left\{\left(1+\sqrt{3}\right)^{n-1} + \left(1-\sqrt{3}\right)^{n-1}\right\} + \frac{1}{2}\left\{\left(1+\sqrt{3}\right)^{n-2} + \cdots \left(1-\sqrt{3}\right)^{n-2}\right\}\right]$$

$$= \left[\left(1+\sqrt{3}\right)^{n-1} + \left(1+\sqrt{3}\right)^{n-2} + \left(1+\sqrt{3}\right)^{n-1} + \left(1+\sqrt{3}\right)^{n-2}\right]$$

$$= \left[\left(1+\sqrt{3}\right)^{n-2}\left(2+\sqrt{3}\right) + \left(1-\sqrt{3}\right)^{n-2}\left(2-\sqrt{3}\right)\right]$$

$$= \left[\left(1+\sqrt{3}\right)^{n-2}\frac{\left(1+\sqrt{3}\right)^2}{2} + \left(1-\sqrt{3}\right)^{n-2}\frac{\left(1-\sqrt{3}\right)^2}{2}\right]$$

$$= \frac{1}{2}\left[\left(1+\sqrt{3}\right)^n + \left(1-\sqrt{3}\right)^n\right]$$

$$Thus, t_n = \frac{1}{2}\left[\left(1+\sqrt{3}\right)^n + \left(1-\sqrt{3}\right)^n\right]$$

So, by the second principle of mathematical induction, the formula is true for all natural numbers.

44. Let T(n) be the statement that $\left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n$ is divisible by 2^n.

T_1: $\left(3+\sqrt{5}\right) + \left(3-\sqrt{5}\right) = 6$ is divisible by $2^1 = 2$ is true.

T_2: $\left(3+\sqrt{5}\right)^2 + \left(3-\sqrt{5}\right)^2 = 28$ is divisible by 2^2 is true. Let us take that T_k is true for all k < n for some n.

T_n: $\left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n$ is divisible by 2^n.

Now, for n −1 < n

$\left(3+\sqrt{5}\right)^{n-1} + \left(3-\sqrt{5}\right)^{n-1}$ is divisible by 2^{n-1}.

$\left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n$

$= \left[\left(3+\sqrt{5}\right)^{n-1} + \left(3-\sqrt{5}\right)^{n-1}\right]\left(3+\sqrt{5}+3-\sqrt{5}\right) - \left[\left(3+\sqrt{5}\right)\left(3-\sqrt{5}\right)^{n-1} + \left(3-\sqrt{5}\right)\left(3+\sqrt{5}\right)^{n-1}\right]$

$= 6\left[\left(3+\sqrt{5}\right)^{n-1} + \left(3-\sqrt{5}\right)^{n-1}\right] - \left[4\left(3-\sqrt{5}\right)^{n-2} + 4\left(3+\sqrt{5}\right)^{n-2}\right]$

$= 3 \times 2\left[\left(3+\sqrt{5}\right)^{n-1} + \left(3-\sqrt{5}\right)^{n-1}\right] - \left[4\left(3+\sqrt{5}\right)^{n-2} + \left(3-\sqrt{5}\right)^{n-2}\right]$

Here $2\left[\left(3+\sqrt{5}\right)^{n-1} + \left(3-\sqrt{5}\right)^{n-1}\right]$ is divisible by $2 \times 2^{n-1} = 2^n, and\ 4\left[\left(3+\sqrt{5}\right)^{n-2} + \left(3-\sqrt{5}\right)^{n-2}\right]$ is divisible by $4 \times 2^{n-2} = 2^n$

Thus $\left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n$ is divisible by 2^n i.e., T_n is true if $t_{n-1}\ and\ t_{n-2}$ are true.

Similarly, $t_1\ and\ t_2$ are true and, therefore, by the second principle of mathematical induction, T_n is true for all $n \in N$.

45. $(xyz)_{11} = (zyx)_{12}$

$11^2x + 11y + z = 9^2z + 9y + x$

$\Rightarrow 120x + 2y - 80z = 0$

$\Rightarrow 60x + y - 40z = 0$

$\Rightarrow 40z - 60x = y$

$\Rightarrow 20(2z - 3x) = y, but\ 0 \le y < 9$

So 20|y, but as $0 \le y < 9, y = 0$

Therefore, $2z = 3x$. As $0 \le x, z < 9$, the solutions are $x = 2, z = 3\ and\ x = 4, z = 6$. Thus two possible solutions are $(203)_{11}\ and\ (406)_{12}$.

46. Let $n = qk + r, 0 \le r \le k$.

$$Now, \frac{n}{k} = \frac{qk+r}{k} = q + \frac{r}{k};\ \frac{n+1}{k} = \frac{qk+r+1}{k} = q + \frac{r+1}{k};$$

$$\frac{2n}{k} = \frac{2qk+2r}{k} = 2q + \frac{2r}{k}; 0 \le r < k$$

Thus, (i) r may be equal to k−1, or

(ii) r may be < k−1

If $r = k - 1$, we have

$$\left[\frac{n}{k}\right] = q. \left[\frac{n+1}{k}\right] = \left[q + \frac{k}{k}\right] = q + 1$$

$$\left[\frac{2n}{k}\right] = \left[2q + \frac{2k-2}{k}\right] = 2q + 1\ \left[since\ k > 1. \frac{2}{k} \le 1\right]$$

So, by adding and equating, we get

$$\left[\frac{n}{k}\right] + \left[\frac{n+1}{k}\right] = 2q + 1 = \left[\frac{2n}{k}\right]$$

(ii)If r < k−1 we have

$$\left[\frac{n}{k}\right] = q. \left[\frac{n+k}{k}\right] = q$$

$$\left[\frac{2n}{k}\right] = \left[2q + \frac{2r}{k}\right] \ge 2q$$

So by adding, we get

$$\left[\frac{n}{k}\right] = \left[\frac{n+1}{k}\right] \le \left[\frac{2n}{k}\right]$$

Combining (i) and (ii), we get

$$\left[\frac{n}{k}\right] + \left[\frac{n+1}{k}\right] \le \left[\frac{2n}{k}\right]$$

Note : When $k = 2$, the above inequality holds as an equality. (verify).

47. We need to find the largest e such that $10^e |\ 6250!$.

But as $10 = 2 \times 5$, this implies that we need to find the largest e such that $5^e | 6250!$ (clearly a larger power of 2| 6250!)

$$But, e = \sum_{i=1}^{\infty} \left[\frac{6250}{5^i}\right] = 1250 + 250 + 50 + 10 + 2 = 1562$$

Hence 6250! Ends with 1562 zeroes.

48. If e is the maximum power of 5 in n!, then

$$e = \sum_{i=1}^{\infty} \left[\frac{n}{5^i}\right] < \sum_{i=1}^{\infty} \left[\frac{n}{5^i}\right] = \frac{n}{5} \sum_{i=1}^{\infty} \frac{1}{5^{i-1}}$$

$$= \frac{\frac{n}{5}}{1 - \frac{1}{5}} = \frac{n}{4}$$

$\therefore n = 4e$

Here e is given to be 20.

$\therefore n \geq 80$. For 80, $e = 19$.

Therefore, 85 is the required answer. Not only 85, 86, 87, 88, 89 are also valid values of n. If solution exists for this type of problem, there will be 5 solutions.

49. $x + y = [x] + [y] + \{x\} + \{y\}$

$[x + y] = [x] + [y] + [\{x\} + \{y\}] \geq [x] + [y]$

This can be generalized for n numbers:

$[x_1] + [x_2] + \cdots + [x_n] \leq [x_1 + x_2 + \cdots + x_n]$

50. $12345 \leq x + 2x + 4x + 8x + 16x + 32x = 63x$

$\therefore x \geq \frac{12345}{63} = 195\frac{20}{21}$

When $x = 196$, the L.H.S of the given equation becomes 12348.

$$\therefore 195\frac{20}{21} \leq x \leq 196$$

Consider x in the interval $\left(195\frac{31}{32}, 196\right)$. The L.H.S. expression of the given equation.

$$= 195 + 0 + 390 + 1 + 780 + 3 + 1560 + 7 + 3120 + 15 + 6240 + 31 = 12342 < 12345$$

When x < 195 $\frac{31}{32}$, the L.H.S. is less than 12342.

$\therefore$ for no value of x. The given equality will be satisfied.

51. If x > 0, then $x^2 < x^2 + 1 + x < x^2 + 2x + 1 = (x + 1)^2$.

So $x^2 + x + 1$ lies between the two consecutives square integers and hence, cannot be a square.

$If\ x = 0, y^2 = 1 + 0 + 0 = 1$ is a square number. Thus, the solutions in this case are is (0, 1), (0, −1).

Again if x < −1, then $x^2 > x^2 + x + 1 > x^2 + 2x + 1$, and hence, there exists no solution.

For $= -1$, we have

$y^2 = 1 - 1 + (-1)^2 = 1$

$\therefore y\ = \pm 1$

Thus, the only integral solutions are (0,1), (0, −1), (−1, +1), (−1, −1)

52. Since $101 \times 10001 \times \ldots \times 1000 \ldots 01$

$= (10^2+1)(10^4+1)(10^8+1)\ldots(10^{2^7}+1)$

$= \left(10^{2^1}+1\right)\left(10^{2^2}+1\right)\ldots(10^{2^7}+1)$

$= (10^2+1)(10^4+1)\ldots(10^{128}+1)$

Multiply and divide by 10^2-1,

$$\frac{(10^2-1)(10^2+1)}{(10^2-1)}(10^4+1)(10^8+1)\ldots(10^{2^7}+1)$$

$$\frac{1}{(10^2-1)}(10^4-1)(10^4+1)(10^8+1)\ldots(10^{2^7}+1)$$

$$= \frac{1}{(10^2-1)}(10^8-1)(10^8+1)\ldots(10^{2^7}+1)$$

$$= \frac{1}{(10^2-1)}(10^{2^8}-1) = \frac{[(10^2)^{128}-1]}{10^2-1}$$

$$= (10^2)^{127} + (10^2)^{126} + \cdots + 1$$

This number has 128, 1's in it with a 0 between every two ones.

$$So, \frac{(10^2)^{128}-1}{99} = \frac{(10^2-1)[(10^2)^{127}+(10^2)^{126}+\cdots+10^2+1}{99}$$

$$= 10^{254} + 10^{252} + \cdots + 10^2 + 1 = 101010\ldots101$$

(These are 128 ones alternating zeroes and there are 127 zeroes in between).

53. $56789 \equiv 89\ (mod\ 100)$

$= -11\ (mod\ 100)$

$\therefore (56789)^{49} \equiv (-11)^{41}$ (mod 100)

$\equiv (-11)^{40} \times (-11)$ (mod 100)

$\equiv 11^{40} \times (-11)$ (mod 100)

$11^2 \equiv 21$ (mod 100)

$11^4 \equiv 41$ (mod 100)

$11^6 \equiv 21 \times 41$ (mod 100)

$\equiv 61$ (mod 100)

$11^{10} \equiv 41 \times 61$ (mod 100)

$\equiv 01$ (mod 100)

$11^{40} \equiv (01)^{40}$ (mod 100)

$\equiv 1$ (mod 100)

$(-11)^{41} \equiv 11^{40} \times (-11)$ (mod 100)

$\equiv 1 \times (-11)$ (mod 100)

$\equiv -11$ (mod 100)

$\equiv 89$ (mod 100)

That is last two digits of $(56789)^{41}$ are 8 and 9 in that order.

54. Let $P = \frac{1.3.5.7\ldots(2n-3).(2n-1)}{2.4.6\ldots(2n-2).2n}$

Here we will prove that the product P_n is actually less than $\frac{1}{\sqrt{3n+1}}$ for n > 1 and greater than $\frac{1}{\sqrt{4n+1}}$.

$$P_1 = \frac{1}{2}, P_2 = \frac{1.3}{2.4} = \frac{3}{8}, P_3 = \frac{1.3.5}{2.4.6} = \frac{15}{48}$$

And writing $n = 1, we\ get\ P_1 = \frac{1}{2}$

$$P_2 = \frac{3}{8} < \frac{1}{\sqrt{3 \times 2 + 1}} = \frac{1}{\sqrt{7}}$$

$$For\ P_2^2 = \frac{9}{64} and \left(\sqrt{\frac{1}{7}}\right)^2 = \frac{1}{7}\ and\ P_2^2 = \frac{9}{24}$$
$$< \frac{1}{7}\ and\ hence\ P_2 < \frac{1}{\sqrt{7}}$$

$n = 3$ gives $\frac{1}{\sqrt{3n+1}} = \frac{1}{\sqrt{10}}$

$$P_3 = \frac{12}{48} < \frac{1}{\sqrt{10}}$$

$$For, P_3^2 = \frac{225}{2304} < \frac{225}{2250} = \frac{1}{10}$$

$$Now\ let\ P_n^2 = \frac{1^2.3^2.5^2 \dots (2n-1)^2}{2^2.4^2.6^2 \dots (2n)^2}$$

We have verified

$$P_1^2 \leq \frac{1}{\sqrt{4}}, P_2^2 < \frac{1}{\sqrt{3 \times 2 + 1}}, P_3^2 < \frac{1}{\sqrt{3 \times 3 + 1}}$$

We use mathematical induction to prove our assertion.

We have verified that for $n = 2,3$

$$P_n < \frac{1}{\sqrt{3n+1}}\ or\ equivalently\ P_n^2 < \frac{1}{3n+1}$$

Let us assume that this result is true is true for $n = m$.

$$i.e., P_m^2 < \frac{1}{3m+1}$$

$$i.e., P_m^2 < \frac{1^2.3^2 \dots (2m-1)^2}{2^2.4^2 \dots (2m)^2} < \frac{1}{3m+1}$$

$$P_{m+1}^2 = \frac{1^2.3^2 \dots (2m-1)^2.(2m+1)^2}{2^2.4^2 \dots (2m)^2.2^2(m+1)^2}$$

$$< \frac{1}{(3m+1)} \times \frac{(2m+1)^2}{2^2(m+1)^2}$$

$$\frac{1}{(3m+1)} \times \frac{(2m+1)^2}{2^2(m+1)^2}$$
$$= \frac{4m^2+4m+1}{4(3m+1)(m^2+2m+1)}$$

$$= \frac{4m^2+4m+1}{12m^3+28m^2+20m+4}$$

$$< \frac{4m^2+4m+1}{12m^3+28m^2+19m+4}$$

Where m is positive

$$= \frac{(4m^2+4m+1)}{(4m^2+4m+1)(3m+4)}$$

$$= \frac{1}{3m+4} = \frac{1}{3(m+1)+1}$$

Thus, $P_m^2 < \frac{1}{3m+1}$ implies $P_{m+1}^2 < \frac{1}{3(m+1)+1}$ and $P_2^2, P_3^2, \dots$ are less than $\frac{1}{3\times2+1}$ and $\frac{1}{3\times3+1}$ respectively.

$$\therefore P_n^2 < \frac{1}{3n+1}\ for\ all\ n > 2$$

$$or, P_n < \frac{1}{\sqrt{3n+1}}$$

In the problem, we have $n = 50$

$$so, \frac{1.3 \dots (2 \times 50 - 1)}{2.4 \dots (2 \times 50)} < \frac{1}{\sqrt{150+1}} = \frac{1}{\sqrt{151}}$$
$$< \frac{1}{\sqrt{100}} = \frac{1}{10}$$

Here we shall show that

$$\frac{1.3.5 \dots (2n-1)}{2.4.6 \dots (2n)} > \frac{1}{\sqrt{4n+1}}$$

$$Let\ P_n = \frac{1.3.5 \dots (2n-1)}{2.4.6 \dots (2n)}$$

$$Then\ P_n^2 = \frac{1^2.3^2.5^2 \dots (2n-1)^2}{2^2.4^2.6^2 \dots (2n)^2}$$

$$P_1^2 = \frac{1^2}{2^2} = \frac{1}{4} > \frac{1}{5}$$

$$Hence\ \ P_1 > \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{4 \times 1 + 1}}$$

So the assumption is true for $n = 1$.

Let us assume that

$$P_m^2 > \frac{1}{4m+1}$$

$$\Rightarrow P_m > \frac{1}{\sqrt{4m+1}}$$

$$P_{m+1}^2 = \frac{1^2.3^2 \dots (2m-1)^2.(2m+1)^2}{2^2.4^2 \dots (2m)^2.(2m+2)^2}$$

$$> \frac{1}{4m+1} \times \frac{(2m+1)^2}{(2m+2)^2}$$

$$But\ \frac{1}{4m+1} \times \frac{(2m+1)^2}{(2m+2)^2}$$

$$= \frac{4m^2+4m+1}{(4m+1)(4m^2+8m+4)}$$

$$= \frac{(4m^2+4m+1)}{16m^3+36m^2+24m+4}$$

$$= \frac{(4m^2+4m+1)}{(4m^2+4m+1)(4m+5)-1}$$

$$> \frac{4m^2+4m+1}{(4m^2+4m+1)(4m+5)} = \frac{1}{(4m+5)}$$

$$So, P_{m+1}^2 \frac{1^2.3^2 \dots (2m-1)^2.(2m+1)^2}{2^2.4^2 \dots (2m)^2.(2m+2)^2}$$

$$> \frac{1}{(4m+1)} \times \frac{(2m+1)^2}{(2m+1)^2} > \frac{1}{4m+5}$$

$$= \frac{1}{4(m+1)+1}$$

$$\therefore P_{m+1} > \frac{1}{\sqrt{4(m+1)+1}}$$

P_1 is true, the truth of P_m implies truth of P_{m+1}. So P_n is true for all $n \in N$.

55.

$2222 \equiv 3$ (mod 7)

$2222^2 \equiv 9 = 2$ (mod 7)

$2222^4 \equiv 4$ (mod 7)

$2222^6 \equiv 8 = 1$ (mod 7)

$2222^{5555} = [(2222)^6]^{925} \times 2222^5$

$= [(2222)^6]^{925} \times 2222^4 \times 2222^1 = 1 \times 4 \times 3$

$= 12 = 5$ (mod 7)

$5555 \equiv 4$ (mod 7)

$5555^3 \equiv 4^3\ (mod\ 7) = 1$(mod 7)

$(5555)^{2222} = (5555^3)^{740} \times 5555^2$

$\equiv 1 \times 4 \times 4$ (mod 7)

$= 2$ (mod 7)

And hence $2222^{5555} + 5555^{2222} \equiv 5 + 2 = 0$ (mod 7) and, hence, the result.

56. $a_1a_2a_3a_4a_5$ is divisible by 5 and hence $a_5 = 5$.

$a_1a_2, a_1a_2a_3a_4, and\ a_1a_2a_3a_4a_5a_6$ are to be divisible by 2, 4 and 6 respectively.
$a_2, a_4\ and\ a_6$ should be even numbers.

So $a_1 = 1$ and $a_3 = 3\ or\ a_1 = 3\ and\ a_3 = 1$

Case 1: If $a_1 = 1, a_2 =$ can be 2, 4 or, 6 and $a_1a_2a_3 = 123, 143$ or 163 but 143, 163 are not divisible by 3.

So $a_1a_2a_3$ should be 123. For a_4, we have either 4 or 6 but for $a_4 = 4$, 1234 is not divisible by 4 and hence $a_4 = 6$ and hence. The six digit number, when $a = 1$, is 123654.

Case 2 : If $a_1 = 3, a_2$ can be2 or 6 or 4 but then $a_1a_2a_3 = 321$ is divisible by 3 and 361 o 341 is not divisible by 3.

So, a_2 can be 6 or 4.

Now $a_1a_2a_3a_4 = 321a_4$ and a_4 can be 4 or 6. For $a_4 = 4$ $321a_4$ is not divisible by 4 and hence $a_4 = 6$ and $a_6 = 4$'

The number, thus, is 321654.

Thus, there are exactly 2 numbers 123654 and 321654 satisfying the condition.

57. It is given that $m \times n = 25! = 2^{22} \times 3^{10} \times 5^6 \times 7^1 \times 11^1 \times 13^1 \times 17^1 \times 19^1 \times 23^1$

Thus 25! Is the product of powers of 9 prime numbers. Thus number of ways in which 25! Can be written as the product of two relatively prime numbers m and n is 2^9, which leads to 2^9 factors, exactly half of which, are less than 1. There are 2^8 such fractions.

58. Since $10^n \equiv 1$ (mod 9) for all $n \in N$, any number written in decimal representation such as $a_n\, a_{n-1}a_{n-2} \dots a_1a_0 \equiv a_n + a_{n-1} + \cdots + a_1 + a_0$ (mod 9).

$\therefore 4333 \equiv 4 + 3 + 3 + 3 \equiv 13\ (mod\ 9) \equiv$ 4(mod 9)

$\therefore 4333^3 \equiv 4^3$ (mod 9)

$\equiv 64$ (mod 9)

$\equiv 1$ (mod 9)

i.e., when 4333^3 is divided by 9, the remainder is 1.

59. Here we should show that there does not exist any positive integer d, which makes (2d−1), (5d−1), (13d−1) to be square number simultaneously.

Assuming the contrary,

$$2d - 1 = x^2$$

$$5d - 1 = y^2$$

$$13d - 1 = z^2$$

Where x, y and z are positive integers $x^2 = 2d - 1$ is an odd number, and since $x^2 \equiv 1$ (mod 8) for all odd integer x if d is even, then d= 2, 4 or 6 (mod 8).

2d−1= 3, 7 or 3 (mod 8) which is impossible and hence d must be odd. Hence y and z are even.

Now $z^2 - y^2 = 8$d

$\Rightarrow (z - y)(z + y) = 8d$

Therefore either (z –y) or (z+y) is divisible by 4.

If z−y is divisible by 4, then z + y $= (z - y) + 2y$ is also divisible by 4 because (z – y) and 2y are divisible by 4.

Similarly, if z + y is divisible by 4, then z – y = (z + y) – 2y is also divisible by 4.

Thus, (z – y). (z + y) is divisible by $4 \times 4 = 16$.

Thus, 16|8d, where d is an odd number.

This is a contradiction and hence, (2d – 1), (5d – 1) and (13d – 1) cannot simultaneously be square integers.

60. We shall make groups of the terms of the expression as follows :

$(1^{1997} + 1996^{1997}) + (2^{1997} + 1995^{1997}) + \cdots + (998^{1997} + 999^{1997})$

Here each bracket is of the form $(a_i^{2n+1} + b_i^{2n+1})$ is divisible by $(a_i + b_i)$

But $(a_i + b_i) = 1997$ for all i.

∴ Each bracket and hence, their sum is divisible by 1997.

61. If possible, let $log_3\, 2$ be a rational number $\frac{p}{q}$ where p, q are integers, $q \neq 0$.

$$log_3^2 \ = \frac{p}{q}$$

$$\Rightarrow 3^{\frac{p}{q}} = 2$$

$$\Rightarrow 3^p = 2^q$$

$3|3^p\ but\ 3\ |2^q\ and\ also\ 2\ |\ 3^p$ and hence, it is a contradiction.

Or 3^p is an odd number and 2^q is an even number but an odd number equals to an even number is a contradiction.

[Note that $3^0 = 1 < 2 = 3^{p/q} < 3^1 = 3$, 0 < p/q < 1 and both p and q are positive real numbers. What we have proved here p/q is not a rational number or there cannot be exist integers satisfying $3^p = 2^q$]

62. Since $x^3 - z^3 = 721$

$\Rightarrow x^3 - z^3 = (x - y)(x^2 + xz + z^2) = 721$

For integral x, z; $x^2 + xz + z^2 > 0$,

$\because x^3 - z^3 = 721$

$\Rightarrow\ x^3 - z^3 > 0$

$\Rightarrow x - z > 0$

So (x – z) $(x^2 + xz + z^2) = 721 = 1 \times 721 = 7 \times 103 = 103 \times 7 = 721 \times 1$

Case (i) $x - z = 1 \Rightarrow x = 1 + z$

And $x^2 + xz + z^2 = (1 + z)^2 + (1 + z)z + z^2 = 721$

$\Rightarrow 3z^2 + 3z - 720 = 0$

$\Rightarrow z^2 + z - 240 = 0$

$\Rightarrow (z + 16)(z - 15) = 0$

$\Rightarrow z = -16$ or 15

Solving, we get

$x = -15\ or\ 16$

So (−15, −16) and (16, 15) is two of the ordered pairs.

Case (ii) $x - z = 7\ or\ x = 7 + z$

And $x^2 + xz + z^2 = 103$

$\Rightarrow (7 + z^2) + (7 + z)z + z^2 = 103$

$\Rightarrow 3z^2 + 21z - 54 = 0$

$\Rightarrow z^2 + 7z - 18 = 0$

$\Rightarrow (z + 9)(z - 2) = 0$

$\Rightarrow z = -9\ or\ z = 2$

So the corresponding values of x are −2 and 9.

So the other ordered pairs are (−2, −9), (9, 2).

Corresponding to x – z = 103 and x – z = 721, the values are imaginary and hence, there are exactly four ordered pairs of integers (−15, −16), (16, 15), (−2, −9) and (9, 2),

satisfying the equation $x^3 = z^3 + 721$.

63. Since $1897 = 7 \times 271$

Now $(2903^n - 803^n) - (464^n - 261^n) =$ $(2903 - 830)|\,(2903^n - 803^n)$

And (464 – 261) | $(464^n - 261^n)$

i.e. $2100|(2903^n - 803^n)$

$203|(464^n - 261^n)$

⇒ 7 | $(2903^n - 803^n)$

And $7|(464^n - 261^n) \because 2100 = 7 \times 300$

$203 \ = 7 \times 29$

Hence, 7 | E

Again, $2903^n - 803^n - 464^n + 261^n$

$= (2903^n - 464^n) - (803^n - 261^n)$

$2903 - 464 = 2439|(2903^n - 464^n)$

And $(803 - 261) = 542|\,(803^n - 261^n)$

i.e. $2439 = 271 \times 9$ | $(2903^n - 464^n)$

and $542 = 271 \times 2$ | $(803^n - 261^n)$

So, 271 | $(2903^n - 464^n)$

And also 271 | $(803^n - 261^n)$

And hence, 271 | E.

Thus, the given expression is divisible by the prime numbers 7 and 271 and, hence is divisible by $271 \times 7 = 1897$.

64. If $p = 2, \frac{2^p-1}{p} = \frac{1}{2}$ is not even an integer.

Let p be a prime of the form 4k + 1.

Then, if $\frac{2^p-1}{p} = \frac{2^{4k}-1}{4k+1} = m^2$ for some odd integer m.

Then $2^k - 1 = (4k+1)m^2$

Since m² is an odd number, $m^2 \equiv |\ (mod\ 4)$ as all odd squares leave a remainder/ when divide by 4 and hence of the form 4l + 1 (say)

Then $2^{4k} - 1 = (4k+1)(4l+1) = 1$ (mod 4)

But the left hand side

$2^{4k} - 1 = (16^k - 1) \equiv -1$ (mod 4)

$\equiv 3$ (mod 4)

And it is a contradiction and hence p cannot be of the form 4k + 1.

So, let p be of the form 4k + 3.

Firstly, let us take $\mathrm{k} = 0, then\ p = 3$.

$so, \frac{2^{p-1}-1}{3} = \frac{2^2-1}{3}$ is a square.

Therefore, $p = 3$ is one of the solutions.

Let p be 4k + 3 with k > 0.

$2^{p-1} - 1 = 2^{4k+1} - 1 = \left(2^{2k+1} - 1\right)(2^{2k+1} + 1)$

And $2^{2k+1} - 1\ and\ 2^{2k+1} + 1$ being consecutive odd numbers are relatively prime.

So, $2^{p-1} - 1 = pm^2$

⇒ $\left(2^{2k+1} - 1\right)\left(2^{2k+1} + 1\right) = (4k+3)m^2 = pm^2$

So, pm^2 could be written as $pu^2 \times v^2$ $where\ pu^2\ and\ v^2$ are relatively prime.

Thus, $2^{2k+1} - 1 = pu^2$

And $2^{2k+1} + 1 = v^2$

$\Rightarrow 2^{2k+1} = v^2 - 1 = (v+1)(v-1)$

So, (v + 1) and (v – 1) are both powers of 2.

Two powers of 2 differ by 2 only if they are 2 and 2^2. In all other cases, the difference will be greater than 2.

So, $v - 1 = 2^1 = 2$

$v + 1 = 2^2 = 4\ \Rightarrow v = 3$

i.e., $2^{2k+1} = 2^3 = 8$

hence, $k = 1\ and\ p = 4k + 3 = 7$.

Therefore, the only other possibility is $p = 7$.

Thus, for $p = 7, \frac{2^p - 1}{p}\ = \frac{2^7 - 1}{7} = \frac{63}{7} = 9$ which is a perfect square.

Thus, the only primes satisfying the given conditions are 3 and 7.

65. From 5!, all the numbers will have the unit digit zero and from 10!, all the unit and tens digit will be zero. So, the unit digit of the numbers S is the unit digit of

1! + 2! + 3! + 4! $= 1 + 2 + 6 + 24 = 33$.

That is unit digit of S, is 3

The tens digit of S, is the tens digit of

1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9!$= 33 + 120 + 720 + 5040 + 40320 + 362880$

So, to get the tens digit of S, add only the tens digit of 33 + 120 + … + 362880 which is 3 + 2 + 2 + 4 + 2 + 8 $= 21$

So, the tens digit of S is 1.

66. N is divisible by 9, if the digit sum is divisible by 9.

The digital sum of N :

The number of 1^S occurring in the digits form 10 to 19 $= 11$

And from 20 to 99 $= 8$.

So total of ones is 11 + 8 $= 19$.

Similarly, no. of $2^S, 3^S$ … 9 are all equal to 19.

So sum of all the digits $= 19(1 + 2 + 3 + \cdots + 9) = \frac{19\times9\times10}{2} = 19 \times 5 \times 9 = 855$.

And hence, 1011 …99 is divisible by 9.

When the number start from 12, the sum of the digits become 855 − 2$= 852$ (Since 10, 11 account for the digital sum 3) and hence, is divisible by 3.

(a) For divisibility by 3, it could start from 13, 15, 16, 18, 19, 21, 22, 24, 25,…
(b) For divisibility by $3^2 = 9$ the numbers may start from any of 18, 19, 27, 28, 36, 37,…

67. Let $2^n - 1 = q$

We have already seen that 1, $d_1 d_2 \dots d_k$ are

$1, 2, 2^2, \dots, 2^{n-1}, q, 2q, \dots, 2^{n-1}q$ respectively.

$$So, S = \frac{1}{1} + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k}$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{q}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}\right]$$

$$\therefore S = \frac{2^n - 1}{2^{n-1}} + \frac{1}{q}\frac{(2^n - 1)}{2^{n-1}} = \frac{(2^n - 1)q + (2^n - 1)}{q2^{n-1}}$$

$$= \frac{(2^n - 1)(q + 1)}{q2^{n-1}} = \frac{(2^n - 1)(2^n)}{(2^n - 1)(2^{n-1})}$$

$$= \frac{2^n}{2^{n-1}} = 2$$

68. The divisors of N are $1, P_1, P_2, P_3, P_1P_2, P_1P_3, P_2P_3, P_1P_2P_3$

It is given that

$1 + P_1 + P_2 + P_3 + P_1P_2 + P_1P_3 + P_2P_3 + P_1P_2P_3 = 3N$

$$Now, \sum_{i=1}^{N} \frac{1}{d_i} = \frac{1}{1} + \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} + \frac{1}{P_1P_2} + \frac{1}{P_1P_3} + \frac{1}{P_2P_3} + \frac{1}{P_1P_2P_3}$$

$$= \frac{P_1P_2P_3 + P_2P_3 + P_1P_3 + P_1P_2 + P_3 + P_2 + P_1 + 1}{P_1P_2P_3}$$

But the numerator is the sum of the divisors of N.

i.e., $\sum_{d|N} d = 3N = 3P_1P_2P_3$ and hence

$$\sum_{i=1}^{N} \frac{1}{d_1} = \frac{3P_1P_2P_3}{P_1P_2P_3} = 3$$

69. If $N = p_1^{a_1}, p_1^{a_2}, \dots p_n^{a_n}$, then the sum of the divisors of N is given by the formula

$$\sum d(N) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \times \frac{p_1^{\alpha_1+2} - 1}{p_n - 1} \times \dots \frac{p_n^{\alpha_n+2} - 1}{p_n - 1}$$

So, the sum of the divisors of $2^n a.b = (2^{n+1} - 1) \times \frac{a^2-1}{a-1} \times \frac{b^2-1}{b-1}$

$$= (2^{n+1} - 1)(a + 1)(b + 1)$$

$$= (2^{n+1} - 1)(9.2^{2n-1})$$

But $2^n ab = 2^n[9.2^{2n-1} - 9.2^{n-1} + 1]$ (on simplification).

The sum of the divisors of $2^n ab$ other than $2^n a.b$ is

$9.2^{2n-1}(2^{n+1} - 1) - 2^n(9.2^{2n-1} - 9.2^{n-1} + 1)$

$= 9.2^{3n} - 9.2^{2n-1} - 9.2^{3n-1} + 9.2^{2n-1} - 2^n$

$= 9.2^{3n-1}(2 - 1) - 2^n$

$= 9.2^{3n-1} - 2^n$

$= 2^n(9.2^{2n-1} - 1)$

$= 2^n.c$

Thus, the sum of the divisors of $2^n.ab$ other than itself is $2^n c$.

Now, sum of the divisors of $2^n c$ other than itself is

$$\frac{2^{n+1} - 1}{2 - 1} \times \frac{c^2 - 1}{c - 1} - 2^n.c$$

$$= (2^{n+1} - 1)(c + 1) - 2^n.c$$

$$= (2^{n+1} - 1)9.2^{2n-1} - 2^n(9.2^{2n} - 1)$$

$$= 9.2^{3n} - 9.2^{2n-1} - 9.2^{3n-1} + 2^n$$

$$= 9.2^{3n-1} - 9.2^{2n-1} + 2^n$$

$$= 2^n[9.2^{2n-1} - 9.2^{2n-1} + 1]$$

$$= 2^n ab$$

i.e., the sum of the divisors of $2^n.c$ other than $2^n.c$ is equal to $2^n ab$.

70. Since $N = 30240 = 2^5 \times 3^3 \times 5^1 \times 7^1$

$$So, s(N) = \frac{(2^6-1)}{2-1} \times \frac{(3^4-1)}{(3-1)} \times \frac{(5^2-1)}{(5-1)} \times \frac{(7^2-1)}{(7-1)}$$

$$= 63 \times 40 \times 6 \times 8$$

$$= 2^7 \times 3^5 \times 5 \times 7$$

$$= 2^2 \times 2^5 \times 3^3 \times 5^1 \times 7^1 = 4 \times N = 4N$$

71. The divisors of $P_1^{a_1}.P_2^{a_2}$ are of the form $P_1^r.P_2^s$ where $0 \le r \le a_1$ *and* $0 \le s \le a_2$.

$$Now, f(P_1^{a_1}.P_2^{a_2}) = \sum_{\substack{0\le r\le a_1 \\ 0\le s\le a_2}} t(P_1^r.P_2^s)$$

$$= \sum_{0\le r\le a_1} \sum_{0\le s\le a_2} (r+1)(s+1)$$

$$= \sum_{0\le \le a_1} (r+1)\left[\sum_{0\le s\le a_1} (s+1)\right]$$

$$\sum_{0\le r\le a_1} (r+1)\left(\frac{(a_2+1)(a_2+2)}{2}\right)$$

$$= \frac{(a_2+1)(a_2+2)}{2} \sum_{0\le r\le a_1} (r+1)$$

$$= \frac{(a_2+1)(a_2+2)}{2}\frac{(a_1+1)(a_1+2)}{2}$$

$$\because f(P_1^{a_1}) = \sum_{0\le r\le a_1} t(P_1^r) = \sum_{0\le r\le a_1} (r+1)$$

$$= \frac{(a_1+1)(a_1+2)}{2}$$

$$Similarly, f(P_2^{a_2}) = \frac{(a_2+1)(a_2+2)}{2}$$

$$\therefore f\left(P_1^{a_1}.P_2^{a_2}\right) = f\left(P_1^{a_1}\right).f(P_2^{a_2})$$

Where $p_1 \neq p_2$ i.e., f is multiplicative.

72. Consider F(18).

Divisors of 18 are 1, 2, 3, 6, 9, 18.

No. of divisors of divisors of 18 are 1, 2, 2, 4, 3, 6.

Sum of the cubes of the number of divisors of 18

$= 1^3 + 2^3 + 2^3 + 4^3 + 3^3 + 6^3 = 324$

Now, $18 = 2^1 \times 3^2$

$F(2^1) = 1^3 + 2^3 = 9$

$F(3^2) = F(9) = 1^3 + 2^3 + 3^3 = 36$

And $F(2) \times F(3^2) = 9 \times 36 = 324 = F(18)$

Thus, F is also multiplicative.

73. Any divisor of $P_1^{a_1}$ *is* P_1^r, *where* $0 \le r \le a_1$

$$F(P_1^{a_1}) = \sum_{r=0}^{a_1} t_3(P_1^r) = \sum_{r=0}^{a_1} (r+1)^3$$

= sum of the first $a_1 + 1$ natural numbers.

$$= \left[\frac{(a_1+1)(a_1+2)}{2}\right]^2$$

$$Similarly, F\left(P_2^{a_2}\right) = \left[\frac{(a_2+1)(a_2+2)}{2}\right]^2$$

$$F\left(P_1^{a_1}.P_2^{a_2}\right) = \sum_{\substack{0\le r\le a_1\\0\le s\le a_2}} t_3(P_1^r.P_2^s)$$

$$= \sum_{r=0}^{a_1}.\sum_{s=0}^{a_1}(r+1)^3(s+1)^3$$

$$= \sum_{r=0}^{a_1}(r+1)^3\left(\sum_{s=0}^{a_2}(s+1)^3\right)$$

$$= \sum_{r=0}^{a_1}(r+1)^3.\left[\frac{(a_2+1)(a_2+2)}{2}\right]^2$$

$$= F\left(P_2^{a_2}\right).\sum_{r=0}^{a_1}(r+1)^3$$

$$= F\left(P_2^{a_2}\right)\left[\frac{(a_1+1)(a_1+2)}{2}\right]^3$$

$$= F\left(P_2^{a_2}\right)F(P_1^{a_1})$$

74. Since $F\left(P_1^{a_1}\right) = 1^3+2^3+3^3+\cdots+(a_1+1)^3$

$\left[f\left(P_1^{a_1}\right)\right]^2 = [1+2+3+\cdots+(a_1+1)]^2 = \left[\frac{(a_1+1)(a_1+2)}{2}\right]^3$

$= 1^3+2^3+\cdots+(a_1+1)^3$

$= F(P_1^{a_1})$

75. The solution is based on the result derived in problem 62.

We can show that $F(N) = f(N)^2$.

If $n = P_1^{a_1}.P_2^{a_2}\dots P_n^{a_n}$, then

$F(n) = F\left(P_1^{a_1}.P_2^{a_2}\dots P_n^{a_n}\right)$ $and\ P_1, P_2, \dots, P_n$ are distinct prime numbers and we have proved earlier that F is a multiplicative.

$$\therefore F(n) = F\left(P_1^{a_1}.P_2^{a_2}\dots P_n^{a_n}\right)$$

$$= F\left(P_1^{a_1}\right).F\left(P_2^{a_2}\right)\dots F\left(P_n^{a_n}\right)$$

$$But, F\left(P_1^{a_1}\right) = 1^3+2^3+\cdots+a_i^3$$

$$= \left[\frac{(a_i+1)(a_i+2)}{2}\right]^3\ for\ all\ i \in N$$

We have

$F(n) =$
$\left[\frac{(a_1+1)(a_1+2)}{2}\right]^2.\left[\frac{(a_2+1)(a_2+2)}{2}\right]^2\dots\left[\frac{(a_n+1)(a_n+2)}{2}\right]^2$

$[(a_1+1)(a_1+2)(a_2+1)(a_2+2)\dots(a_n+1)(a_n+2)]^2/(2^n)^2$ ....(A)

Now, $F(n) = f\left(P_1^{a_1}.P_2^{a_2}\dots P_n^{a_n}\right)$

$= f\left(P_1^{a_1}\right).f\left(P_2^{a_2}\right)\dots f\left(P_n^{a_n}\right)$ [∵ f is multiplicative]

$$= \frac{(a_1+1)(a_1+2)}{2}.\frac{(a_2+1)(a_2+2)}{2}$$

$$\dots\frac{(a_n+1)(a_n+2)}{2}$$

$= (a_1+1)(a_1+2)(a_2+1)(a_2+2)\dots(a_n+1)(a_n+2)/2^n$ ....(B)

∴ From (A) and (B), we see that $F(n) = [f(n)]^2$

76. Let $n^2+96 = k^2, where\ k\ \in N.$

Then $k^2-n^2 = 96$

$(k-n)(k+n) = 96 = 3^1\times 2^5$

Clearly k > n and hence, k + n > k – n > 0.

Since 3 is the only odd factor, both k and n are integers.

We must have k + n and k – n both to be either even or odd. (If one is odd and the other even, then k and n do not have integer solutions). Also both k + n and k – n cannot be odd as the product is given to be even. So the difference possibilities for k + n, k – n are as follows.

$$k-n=2 \qquad k+n=48 \ldots .(1)$$

$$k-n=4 \qquad k+n=24 \ldots .(2)$$

$$k-n=6 \qquad k+n=16 \ldots .(3)$$

$$k-n=8 \qquad k+n=12 \ldots .(4)$$

So, solving separately eqns. (1), (2), (3) and (4), we get $n = 23{,}10{,}5{,}2$.

So, there are exactly four values for which n^2+96 is a perfect square.

$n = 23\ gives\ 23^2 + 96 = 625 = 25^2$

$n = 10\ gives\ 10^2 + 96 = 196 = 14^2$

$n = 5\ gi\ ves\ 5^2 + 96 = 121 = 11^2$

$n = 2\ gives\ 2^2 + 96 = 100 = 10^2$

77. Let us write the sequence of the number of beads in the 1st, 2nd, 3rd ,…, nth necklaces.

$= 5, 7, 10, 14, 19\ldots$

$$= (4+1), (4+3), (4+6), (4+10), (4 + 15), \ldots, \left[4 + \frac{n(n+1)}{2}\right]$$

S_n = total number of beads in the n necklaces

$$S_n = \underbrace{(4+4+\cdots+)}_{n\ times} + 1 + 3 + 6 + \cdots + \frac{n(n+1)}{2}$$

= 4n + Sum of the first n triangular numbers.

$$= 4n + \frac{1}{2}\left(\sum n^2 + n\right)$$

$$= 4n + \frac{1}{2}\left(\sum n^2 + \sum n\right)$$

$$= 4n + \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{1}{2}\frac{n(n+2)}{2}$$

$$= 4n + \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}$$

$$= \frac{1}{12}[48n + 2n(n+1)(n+2)]$$

$$= \frac{n}{6}[n^2 + 3n + 26]$$

78. $x_1 = 0$

$= x_2 = 0^2 - i = -i$

$x_3 = (-i)^2 - i = -1 - i = -(1+i)$

$x_4 = [-(1+i)^2] - i = 2i - i = i$

$x_5 = (i)^2 - i = -1 - i = x_3$

$x_6 = (-1-i)^2 - i = i = x_4$

$\therefore x_6 = x_4$ and hence $x_7 = x_5$ and so on

$x_{2n} = 1\ for\ n \geq 1, x_{2n+1} = -1 - i$

So $x_{2000} = i = (0,1)$ in the complex plane.

$x_{1997} = (-1, -i) = (-1, -1)$ in the complex plane.

So, the distance between $x_{2000}\ and\ x_{1997}\ is\ \sqrt{1^2 + 2^2} = \sqrt{5}$.

79. You know that the greatest power of a > 1, $a \in N$, dividing n is given by

$$\sum_{i=1}^{\infty}\left[\frac{n}{a^i}\right] \dots \dots (1)$$

$$But \sum_{i=1}^{\infty}\left[\frac{n}{a^i}\right] < \sum_{i=1}^{\infty}\frac{n}{a^i} = n\left(\frac{1}{a-1}\right) \dots \dots .(2)$$

We want to find n, such that

$$\sum_{i=1}^{\infty}\left[\frac{n}{5^i}\right] = 1998$$

$$By\ eq.(2), \sum_{i=1}^{\infty}\left[\frac{n}{5^i}\right] < n\left(\frac{1}{5-1}\right) = \frac{n}{4}$$

$$So, \frac{n}{4} > 1998$$

n > 7992

By trial and error, we take $n = 7995$ and then search for the correct value. If $n = 7995$, then the number of zeroes at the end of 7995 is by eq. (1).

$$\frac{7995}{5} + \frac{7995}{5^2} + \cdots$$

$$= 1599 + 319 + 63 + 12 + 2 = 1995$$

So true for $n = 8000$, we get the number of zeroes at the end of 8000! = 1600 + 320 + 64 + 12 + 2= 1998.

Note : Corresponding to 1997 zeroes at the end, there exists no n, as 7995! Has 1995 zeroes and the next multiple of 5, i.e., 8000 is a multiple of 125, it adds 3 more zeroes to 1995 given 1998 zeroes at the 9 end.

80. $f(1) = 1$

$f(1) + 2f(2) = 2(2+1)f(2)$

$\Rightarrow 4f(2) = 1, \Rightarrow f(2) = \frac{1}{4}$.

Again, f(1) + 2f(2) + 3f(3)= $(3 \times 4)f(3)$

$$\Rightarrow 9f\ (3) = 1 + \frac{1}{2} = \frac{3}{2}$$

$$\Rightarrow f(3) = \frac{1}{6}$$

The above calculations suggest that f(n) may be 1/2n for n > 1. Let us verify if it is so.

$$For\ n = 2, \quad f(2) = \frac{1}{2 \times 2} = \frac{1}{4}\ is\ true.$$

$$n = 3, \quad f(3) = \frac{1}{3 \times 2} = \frac{1}{6} is\ also\ true.$$

So, let us assume that f(n) = $\frac{1}{2n}$.

Now, we should show that f(n+1) = $\frac{1}{2(n+1)}$.

(Here we use the principle of Mathematical induction).

By the (ii) hypothesis, we have

$f(1) + 2f(2) + \cdots + n(fn) = n(n+1)f(n)$

$f(1) + 2f(2) + \cdots + nf(n) + (n+1)f(n+1) = (n+1)(n+2)f(n+1)$

$$\underbrace{= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{n\ times} + (n+1)f(n+1)$$

$$= (n+1)(n+2)f(n+1)$$

$$\Rightarrow 1 + (n-1)\frac{1}{2} = (n+1)f(n+1)(n+2-1)$$

$$\Rightarrow (n+1)^2 \times f(n+1)$$

$$= f(n+1) = \frac{1+(n-1)\frac{1}{2}}{(n+1)^2} = \frac{n+1}{2(n+1)^2}$$
$$= \frac{1}{2(n+1)}$$

Thus, by the principle of mathematical induction, we have proved that f(n) = $\frac{1}{2n}$ for n > 1.

$$\therefore f(1997) = \frac{1}{2 \times 1997} = \frac{1}{3994}$$

81. $f(2) = 2$

$f(4) = f(2.2) = f(2). f(2) = 2.2 = 4$

$f(8) = f(2.4) = f(2). f(4) = 2.4 = 8$

Thus, we infer that $f(2^n) = 2^n$.

Let us use M. I. for proving

$f(2^1) = 2$ (by hypothesis) ……..(1)

Assume $f(2^n) = 2^n$ ………..(2)

$f(2^{n+1}) \ = f(2.2^n) = f(2). f(2^n) = 2.2^n$ ……..(3)

By hypothesis and eq. (1) and (2), we need to find f(n) for all n. Let us see what happens for f(1), f(3) at first.

f(1) < f(2) (Given)

Now $f(2) = f(1 \times 2) = f(1) \times f(2)$

$\Rightarrow f(1) = 1$

Similarly, f(2) < f(3) < f(4)

2 < f(3) < 4

But the only integer lying between 2 and 4 is 3. Thus f(3) = 3.

So again we guess that $f(n) = n$, for all n.

Let us prove by using the strong principle of mathematical induction.

Let f(n) = n for all n < a, fixed $m \in N$.

Now, we should prove that f(m)= m.

If m is an even integer, then f(m) = $2k$ and k < m.

So, $f(m) = f(2k) = f(2) \times f(k) = 2 \times k = 2k = m$

So, all even m, f(m) = m.

If m is an odd integer, let m = $2k + 1$

And f(2k) < f(2k + 1)< f(2k + 2)

2k < f(2k + 1) < (2k + 1)

(Because the function f(n) = n is true for all even integer n).

But only integer lying between 2k and 2k + 2, is 2k + 1, (since the range of f is integer).

Thus, f(2k + 1) = 2k + 1

i.e., f(m) = m, in the case of odd m also.

Thus, f(n) = n, for all $n \ \in N$

$$\therefore f(1983) = 1983$$

82. First let us show that the expression

$f(m) = \frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{2m+1} + \left(3 - 2\sqrt{2}\right)^{2m+1} - 6\right]$ is an integer.

$$For\ m = \frac{1}{8} \times \left[2 \times 3_{c_0} \times 3^3 + 2 \times 3_{c_2} \times 3^1 \times \left(2\sqrt{2}\right)^2 - 6\right]$$

$$= \frac{1}{8} \times [54 + 144 - 6] = \frac{1}{8} \times [192] = 24$$

And hence, is an integer.

For any m > 1, let us prove that the expression

$$f(m+1) = \frac{1}{8}\Big[\left(3+2\sqrt{2}\right)^{2m+1} + \left(3-2\sqrt{2}\right)^{2m+1} - 6\Big]$$

is an integer. Expanding and cancelling the terms, we get

$$f(m+1) = \frac{1}{8} \times \Big[\left(3+2\sqrt{2}\right)^{2m+1} + \left(3-2\sqrt{2}\right)^{2m+1} - 6\Big]$$

$$= \frac{1}{4} \times \Big[3^{2m+1} + 2m + 1_{c_2}.\,3^{2m-1}.\left(2\sqrt{2}\right)^2 + 2m + 1_{c_4}.\,3^{2m-3}\left(2\sqrt{2}\right)^4 + \cdots + 2m + 1_{c_m}.\,3.\left(2\sqrt{2}\right)^{2m-1} - 3\Big]$$

$$= \frac{1}{4} \times \Big[2m + 1_{c_2}3^{2m+1}\left(2\sqrt{2}\right)^2 + 2m + 1_{c_4}.\,3^{2m-3}.\left(2\sqrt{2}\right)^4 + \cdots + 2m + 1_{c_{2m}}3.\left(2.\sqrt{2}\right)^{2m} + 3^{2m+1} - 3\Big]$$

All the terms in the above expression except $3^{2m+1} - 3$ are multiples of 4, as the even powers of $(2\sqrt{2})$ is a multiple of 4.

$3^{2m+1} - 3 = 3[9^m - 1]$ is also a multiple of 4.

$$Now, f(m) + 1 = \frac{1}{8} \times \Big[\left(3+2\sqrt{2}\right)^{2m+1} + \left(3-2\sqrt{2}\right)^{2m+1} - 6\Big] + 1$$

$$= \frac{1}{8} \times \Big[\left(3+2\sqrt{2}\right)^{2m+1} + \left(3-2\sqrt{2}\right)^{2m+1} - 6 + 8\Big]$$

$$= \frac{1}{8} \times \Big[\left(3+2\sqrt{2}\right)^{2m+1} + \left(3-2\sqrt{2}\right)^{2m+1} + 2\Big]$$

Now, $3 + 2\sqrt{2} = \left(1+\sqrt{2}\right)^2$ and $3 - 2\sqrt{2} = \left(1-\sqrt{2}\right)^2$.

$$So, \frac{1}{8} \times \Big[\left(3+2\sqrt{2}\right)^{2m+1} + \left(3-2\sqrt{2}\right)^{2m+1} + 2\Big]$$

$$= \frac{1}{8} \times \Big[\left\{\left(1+\sqrt{2}\right)^2\right\}^{2m+1} + \left\{\left(1-\sqrt{2}\right)^2\right\}^{2m+1} + 2\Big]$$

$$= \frac{1}{8} \times \Big[\left\{\left(1+\sqrt{2}\right)^{2m+1}\right\}^2 + \left\{\left(1-\sqrt{2}\right)^{2m+1}\right\}^2 + 2\Big]$$

$$= \frac{1}{8} \times \Big[\left\{\left(1+\sqrt{2}\right)^{2m+1}\right\}^2 + \left\{\left(1-\sqrt{2}\right)^{2m+1}\right\}^2 - 2(-1)\Big]$$

$$= \frac{1}{8} \times \Big[\left\{\left(1+\sqrt{2}\right)^{2m+1}\right\}^2 + \left\{\left(1-\sqrt{2}\right)^{2m+1}\right\}^2 - 2 \times \left(1+\sqrt{2}\right)^{2m+1}\left(1-\sqrt{2}\right)^{2m+1}\Big]$$

Since $\left(1+\sqrt{2}\right)^{2m+1}\left(1-\sqrt{2}\right)^{2m+1} = \left[\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right)\right]^{2m+1} = (-1)^{2m+1} = -1.$

So the given expression is equal to

$$= \left\{\frac{\left(1+\sqrt{2}\right)^{2m+1} - \left(1-\sqrt{2}\right)^{2m+1}}{2\sqrt{2}}\right\}^2$$

Note that

$$\frac{(1+\sqrt{2})^{2m+1}-(1-\sqrt{2})^{2m+1}}{2\sqrt{2}}$$

Is an integer, as all the left over terms contain $2\sqrt{2}$ as a factor in the numerator.

$$Now, 2f(m)+1 = \frac{1}{4} \times \left[(3+2\sqrt{2})^{2m+1} + (3-2\sqrt{2})^{2m+1} - 6\right] + 1$$

$$= \frac{1}{4} \times \left[(3+2\sqrt{2})^{2m+1} + (3-2\sqrt{2})^{2m+1} - 2\right]$$

Since n is shown to be an integer, so (2n + 1) is also an integer.

Now, (2n + 1) can be written as

$$= \frac{1}{4} \times \left[\left\{(1+\sqrt{2})^{2m+1}\right\}^2 + \left\{(1-\sqrt{2})^{2m+1}\right\}^2 - 2\right]$$

$$= \frac{1}{4} \times \left[\left\{(1+\sqrt{2})^{2m+1}\right\}^2 + \left\{(1-\sqrt{2})^{2m+1}\right\}^2 + 2 \times \left\{(1+\sqrt{2})(1-\sqrt{2})\right\}^{2m+1}\right]$$

$$= \left\{\frac{(1+\sqrt{2})^{2m+1} + (1-\sqrt{2})^{2m+1}}{2}\right\}^2$$

By a similar reasoning, the expression

$$\frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m+1}}{2}$$

Is an integer. Hence, the result.

83. As the terms containing $\sqrt{2}$ vanishes in the expansion of $(17+12\sqrt{2})^m + (17-12\sqrt{2})^m$ and integral terms are all multiples of 8 and hence, n is an integer. (Prove it).

$$n-1 = \frac{1}{8} \times \left[(17+12\sqrt{2})^m + (17-12\sqrt{2})^m + 6 - 8\right]$$

$$= \frac{1}{8} \times \left[(17+12\sqrt{2})^m + (17-12\sqrt{2})^m - 2\right]$$

We know

$$17+12\sqrt{2} = (3+2\sqrt{2})^2$$

$$17-12\sqrt{2} = (3-2\sqrt{2})^2$$

Again both $(17+12\sqrt{2})(17-12\sqrt{2})$ *and* $(3+2\sqrt{2})(3-2\sqrt{2})$ are equal to 1.

$$So, \frac{1}{8} \times \left[(17+12\sqrt{2})^m + (17-12\sqrt{2})^m - 2\right]$$

$$= \frac{1}{8} \times \left[\left[(3+2\sqrt{2})^m\right]^2 + \left[(3-2\sqrt{2})^m\right]^2 - 2 \times (3+2\sqrt{2})(3-2\sqrt{2})\right.$$

$$= \frac{1}{8} \times \left[\frac{(3+2\sqrt{2})^m - (3-2\sqrt{2})^m}{2\sqrt{2}}\right]^2$$

$$and\ 2n-1 = \frac{1}{4} \times \left[(17+12\sqrt{2})^m + (17-12\sqrt{2})^m + 6 - 4\right]$$

$$= \frac{1}{4} \times \left[(17+12\sqrt{2})^m + (17-12\sqrt{2})^m + 2\right]$$

$$= \left[\frac{(3+2\sqrt{2})^m + (3-2\sqrt{2})^m}{2}\right]^2$$

And hence the result.

(Show that $(3+2\sqrt{2})^m - (3-2\sqrt{2})^m$ *and* $\frac{(3+2\sqrt{2})^m+(3-2\sqrt{2})^m}{2}$ *are and so* $\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}$ is also an integer and hence, their sum is also an integer.

Thus, $\frac{1}{32}\left[\left(17+12\sqrt{2}\right)^n+\left(17-12\sqrt{2}\right)^n-2\right]$
is a square integer.

To show that exp. (1) can be written as
$\frac{1}{2}m(m+1)$. Consider the exp. (2)

$$\frac{1}{32}\times\left[\left(17+12\sqrt{2}\right)^n-\left(17-12\sqrt{2}\right)^n-2\right]$$

$$=\left\{\frac{\left(1+\sqrt{2}\right)^n+\left(1-\sqrt{2}\right)^n}{2}\right\}^2\left\{\frac{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n}{2\sqrt{2}}\right\}^2$$

$$=\frac{1}{2}\left[\frac{\left\{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n\right\}^2}{4}\right]\left[\frac{\left\{\left(1+\sqrt{2}\right)^n+\left(1-\sqrt{2}\right)^n\right\}^2}{4}\right]$$

For all n, we shall show that

$$\frac{\left\{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n\right\}^2}{4}\cdot\frac{\left\{\left(1+\sqrt{2}\right)^n+\left(1-\sqrt{2}\right)^n\right\}^2}{4}$$

Are consecutive integers.

Clearly for $n=1$, we get

$$\frac{\left\{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n\right\}^2}{4}=\frac{8}{4}=2$$

$$and\ \frac{\left\{\left(1+\sqrt{2}\right)^n+\left(1-\sqrt{2}\right)^n\right\}^2}{4}=\frac{4}{4}=1$$

And hence for $n=1$,

$$\frac{\left\{\left(1+\sqrt{2}\right)^n+\left(1-\sqrt{2}\right)^n\right\}^2}{4}\ and\ \frac{\left\{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n\right\}^2}{4}$$

Are consecutive integers.

For any n,

$$\frac{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n}{2}\ and\ \frac{\left(1+\sqrt{2}\right)^n+\left(1-\sqrt{2}\right)^n}{2}$$

Are integers (prove) and, hence

$$\left\{\frac{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n}{2}\right\}^2$$
$$=\frac{\left\{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n\right\}^2}{4}$$

$and\ \left\{\frac{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n}{2}\right\}^2=$
$\frac{\left\{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n\right\}^2}{4}$ are integers.

$$Now,\frac{\left\{\left(1+\sqrt{2}\right)^n+\left(1-\sqrt{2}\right)^n\right\}^2}{4}$$
$$=\frac{\left(1+\sqrt{2}\right)^{2n}+\left(1-\sqrt{2}\right)^{2n}-2}{4}$$

$$=\frac{\left(3+2\sqrt{2}\right)^n+\left(3-2\sqrt{2}\right)^n-2}{4}\ldots\ldots(3)$$

$$and\ similarly,\frac{\left\{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n\right\}^2}{4}$$
$$=\frac{\left(3+2\sqrt{2}\right)^n+\left(3-2\sqrt{2}\right)^n-2}{4}\ldots(4)$$

∴ From exp. (3) and exp. (4), we find that

$$\frac{\left\{\left(1+\sqrt{2}\right)^n+\left(1-\sqrt{2}\right)^n\right\}}{4}$$

$$and\ \frac{\left\{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n\right\}}{4}$$

Are integers of the form
$\frac{2k-2}{4}\ and\ \frac{2k+1}{2}\ or\ \frac{1}{2}(k-1)\ and\ \frac{1}{2}(k+1)$

And hence, they differ by $\frac{1}{2}(k+1)-\frac{1}{2}(k-1)=1$.

$$So,\frac{1}{32}\times\left\{\left(17+12\sqrt{2}\right)^n-\left(17+12\sqrt{2}\right)^n-2\right\}$$

$$= \frac{1}{2} \times \frac{\left\{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right\}^2}{4} \times \frac{\left\{\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n\right\}^2}{4}$$

$$= \frac{1}{2}\frac{(k-1)}{2} \times \frac{(k+1)}{2}$$

Or $\frac{1}{2}(m-1)m$ *or equivalently* $\frac{m(m+1)}{2}$ and hence, the result.

Note : This $\frac{1}{32} \times \left[\left(17+12\sqrt{2}\right)^n + \left(17-12\sqrt{2}\right)^n - 2\right]$ gives you an infinite family of square and triangular numbers.

84. Given $a_1 = \frac{1}{2}$ $for\ n \geq 2$

$$So, a_k = \frac{2k-3}{2k}\ a_{k-1}\ for\ k \geq 2$$

Or, $2ka_k = (2k-3)a_{k-1}$

$\Rightarrow 2ka_k - (2k-3)a_{k-1} = 0$

$\Rightarrow 2ka_k - 2(k-1)a_{k-1} + a_{k-1} = 0$

$\Rightarrow 2ka_k - 2(k-1)a_{k-1} = -a_{k-1}$ ....(1)

Now adding up to eq. (1) from $k = 2\ to\ k = (n+1)$, we have

$$\left.\begin{aligned} 4a_2 - 2a_1 &= -a_1 \\ 6a_3 - 4a_2 &= -a_2 \\ 8a_4 - 6a_3 &= -a_3 \\ \vdots \quad \vdots \quad &\vdots \end{aligned}\right\} \ldots\ldots(2)$$

$$2na_n - 2(n-1)a_{n-1} = -a_{n-1}$$

$$2(n+1)a_{n+1} - 2na_n = -a_n$$

Summing, eq. (2), we get

$$2(n+1)a_{n+1} - 2a_1 = -\sum_{k=1}^{n} a_k$$

$$\Rightarrow \sum_{k=1}^{n} a_k = 2a_1 - 2(n+1)a_{n+1}$$

$$= 1 - 2(n+1)a_{n+1}$$

$$a_1 = \frac{1}{2}, a_n = \left(1 - \frac{3}{2n}\right)a_{n-1}$$

$$\Rightarrow a_2 = \left(1-\frac{3}{4}\right)\frac{1}{2} = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

$$\Rightarrow a_3 = \left(1-\frac{3}{6}\right)\frac{1}{8} = \frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$$

$$a_n = \left(1 - \frac{3}{2n}\right)a_{n-1}$$

Is positive as $\left(1-\frac{3}{2n}\right)$ for all n > 2 is positive and $a_1, a_2, a_3, \ldots$ are all positive since each a_i is a product of $\left(1-\frac{3}{2i}\right)$ $a_i - 1$ and $a_1 > 0$ implies that $a_2 > 0, \ldots., a_i - 1 > 0$ and hence

$$\sum_{k=1}^{n} a_k = 1 - 2(n+1)a_{n+1} < 1$$

$$[\because 2(n+1)a_{n+1} > 0]$$

85. If (a, b, c) is a valid triplet then (7 – c, 7 – b, 7 – a) is also a valid triplet as $1 \leq (7-c) \leq (7-b) \leq (7-a) \leq 6$.

Note $(7-b) \neq b$ etc

$$Let\ S = \sum_{1 \leq a \leq b \leq c \leq 6} (a,b,c)\,, then\ by\ the\ above$$

$$S = \sum_{1 \leq a \leq b \leq c \leq 6} (7-a)(7-b)(7-c)$$

$$\therefore 2S = \sum_{1 \leq a \leq b \leq c \leq 6} [(a,b,c) + (7-a)(7-b)(7-c)]$$

$$= \sum_{1 \le a \le b \le c \le 6} [7^3 - 7^2(a+b+c) + 7(ab+bc+\cdots+ca)]$$

In the R.H.S., every term is divisible by 7, i.e., 7 | 2S, and hence, 7 | S.

86. Let the required number be ….abc 7. Since it is given that

$5(\ldots abc7) = 7\ldots abc$

We find that $c = 5$. Putting this value of c back in the equation we have 5(…ab57) = 7…ab5 we give $b = 8$. Continuing this way till we get 7 for the first time, we find that required number is 142857.

87. To test the divisibility of the number N = 19202122… 919293 by 3 or 9 we should find the sum of the digits of N. Noting that 1 occurs 9 times in the digits from 19 to 93 (in 19, 21, 31, …, 91), 2 occurs 18 times (in 20, 21, 22, …, 29, 32, 42, …., 92) etc. we find the sum of the digits of N to be 717. This number is divisible by 3 (since 7 + 1 + 7= 15 is so) but not by 9. Thus the highest power of 3 dividing N is 3.

88. First note that the set of primes dividing x is the same as the set of primes dividing y. Take any prime p dividing x (and hence y also) and suppose it occur to the power α in x and β in y (that is, p^α is the maximum power of p dividing x and p^β is the maximum power of p dividing y). Then

$$x^a = y^b \Rightarrow p^{\alpha a} = p^{\beta b} \Rightarrow \alpha a = \beta b$$
$$\Rightarrow a|\beta b \text{ and } b|\alpha a$$

$\Rightarrow a|\beta \text{ and } b|\alpha \text{ since } (\alpha, \beta) = 1.$

Write $\beta = \alpha\beta_p \text{ and } \alpha = \beta\alpha_p$. Then

$$p^{\alpha a} = p^{\beta b} \Rightarrow p^{ab\alpha_p} = p^{ab\beta_p}$$

$\Rightarrow \alpha_p = \beta_p.$

For each prime p dividing x and (and hence y) get the integer α_p. Verify that the integer

$n = \prod_{p|n} p^{\alpha_p}$ (this notation means n is the product of the numbers p^{α_p} for each prime p dividing n) satisfies the required properties.

89. Let $n = aabb$ be a number satisfying the given properties.

Since n is α square the only possibilities for b are 1, 4, 5, 6 or 9. Among them 1, 5, 6 and 9 are not possible since the numbers aa 11, aa 55, and aa99 leave remainder 3 and aa 66 leaves remainder 2 when divided by 4, which is not possible if n is a square. So b can only be 4. Clearly 11 divides $n = aabb$. Since n is a square and 11 is a prime. 11^2 also divides $n = 11 \times a0b$ that is, 11 divides a0b which implies 11 divides a + b. Since b can be only 4, the only possibility for a is $a = 7$.

Noting that $7744 = (88)^2$ is indeed a square, we conclude that 7744 is the only number with the given properties.

90. Let $[\sqrt{n}] = k$. Then $k^2 < n < (k+1)^2$. Also since k^3 divides n^2, we have that k^2 divides n^2 and hence k divides n.

Thus, the only possibilities for n are $n = k^2 + k$ and $n = k^2 + 2k$.

(i) Let $n = k^2 + k$. Then

$$k^3|n^2 \Rightarrow k^3|(k^2+k)^2 = k^4+2k^3+k^2$$
$$\Rightarrow k^3|k^2 \ \Rightarrow k=1$$

i.e., $n = 2$.

(ii) Let $n = k^2 + 2k$. Then

$$k^3|\, n^2 \ \Rightarrow k^3 \ |(k^2+2k)^2 = k^4+4k^3+4k^2$$

Which implies that $k^3|\ 4k^2 \ or\ k|4$. Therefore, $k = 1, 2\ or\ 4$.

When $k = 1, 2, 4$, we get the corresponding values 3, 8 and 24 for n. Thus, n= $2, 3, 8$ and 24 are all positive integers satisfying the given conditions.

91. Since the product of k consecutive integers is divisible by k!, A(n) is an integer. We compare the highest powers of 2 dividing the numerator and denominator to determine the nature of A(n).

Suppose we express n in the base 2, say,

$n = a_l 2^l + a_{l-1}2^{l-1} + a_{l-2}2^{l-2} + \cdots + a_l.2 + a_0 = 1.$

The highest power of 2 dividing n! is given by

$$s = \left[\frac{n}{2}\right] + \left[\frac{n}{2^2}\right] + \left[\frac{n}{2^3}\right] + \cdots + \left[\frac{n}{2^l}\right]$$

Where [x] denotes the largest integer smaller than x.

But, for $1 \ \leq m \leq l.$

$$\left[\frac{n}{2^m}\right] = a_l 2^{l-m} + a_{l-2}2^{l-m-1} + \cdots + a_m$$

Thus,

$$s = \sum_{m=1}^{l} \left[\frac{n}{2^m}\right]$$

$$= \sum_{m=1}^{l} \sum_{k=m}^{l} a_k 2^{k-m}$$

$$= \sum_{k=1}^{l} a_k \left(\sum_{m=1}^{k} 2^m - 1 \right)$$

$$= \sum_{k=1}^{l} a_k \left(2^k - 1\right)$$

$$= \sum_{k=0}^{l} a_k 2^k - \sum_{k=0}^{l} a_k$$

$$= n - \{sum\ of\ the\ digits\ of\ n\ in\ the\ base\ 2\}$$

Hence the highest power of 2 dividing $(n!)^2$ is $2s = 2n - 2$ (sum of the digits of n in the base 2). Similarly the highest power of 2 dividing (2n)! is t $= 2n -$ (sum of the digits of 2n in the base 2). But the digits of n in base 2 and those of 2n in base 2 are the same except for a zero at the end of the representation for 2n.

Thus

$$t - 2s = a_l + a_{l-1} + a_{l-2} + \cdots + a_l + a_0$$

Where the a_i are the digits of n in base 2. Note that $a_l = 1$. Hence $t - 2s \geq 1$. But A(n) is even if and only if $t - 2s \geq 1$. Hence it follows that A(n) is even for all n.

Moreover A(n) is divisible by 4 if and only if $t - 2s \geq 2$. Since A(1) = 2, 4 does not divide A(1). Suppose n = 2 for some l.

Then $a_l = 1$ and $a_i = 0\ for\ 0 \leq i \leq l-1$. Hence $t - 2s = 1$ and A(n) is not divisible by 4. On the other hand if n is not a power of 2, then for some l,

$$n = 2^l + a_{l-1}2^{l-1} + a_{l-2}2^{l-2} + \cdots + a_0$$

Where $a_i \neq 0$ for at least one i and hence must be equal to 1. Thus $t - 2s \geq 1 + a_i \geq 2$. It follows that A(n) is divisible by 4 if and only if n is not a power of 2.

Remark : Given any prime p, the highest power of s of p dividing n! is given by

$$s = \frac{n - (sum\ of\ the\ digits\ of\ n\ in\ the\ base\ p)}{p-1}$$

92. Take $a = 2^n + \frac{1}{2}, b = 2^{n+1} + \frac{1}{2}$. In the binomial expansions of $a^k\ and\ b^k, 1 \leq k \leq n$, we see that all the terms except the last are integral and the last terms are each equal to 1/2[k]. Hence $a^k - b^k$ is an integer for $1 \leq k \leq n$.

93. ABCD is a cyclic quadrilateral. 'O' is the circumcentre of ΔAPB. That is, if M is the midpoint of PB, then OM is ⊥r to PB in the fig., H, is the ortho−centre of ΔCPD. Let OP produced meet DC in L.

To prove that : O, P and H, are collinear.

That is, to prove that H lies on OP or OP produced.

That is, to prove that H lies on OP or OP produced.

Or, in other words, OP produced is perpendicular to DC.

Proof : Since quadrilateral ABCD is cyclic.

$\angle CDB = \angle CAB = \angle PAB$

$= \frac{1}{2}\angle POB\ (Since\ O\ is\ the\ circumcentre\ of\ \Delta PAB)$

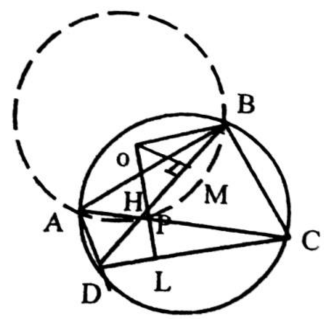
$= \angle POM (= \angle BOM) as\ OM\ is\ the\ \perp r\ bisector\ of\ PB$

In $\Delta LDP\ and\ \Delta MOP$

$\angle LDP = \angle POM$

$\angle DPL = \angle OPM$ $\quad (Vertically\ opp. \angle s)$

$\therefore \angle PLD = \angle PMO = 90°$ and hence the result.



94. Let $\angle ABC = \angle ACB = b°$

AT is the angle bisector of $\angle A$. I is the midpoint of PQ. Now $AP = AQ$ as the smaller circle touches AB and AC at P and Q respectively. The centre of the circle PQT lies on the angle bisector of $\angle A$, namely AT; since PQ is the chord of contact of the circle PQT, $PQ \perp AT$ and the midpoint I of PQ lies on AT.

Now, to prove that I is the incentre of ΔABC, it is enough to prove that BI is the angle bisector of $\angle B$ and CI is the angle bisector of $\angle C$ respectively. By symmetry, $\angle PTI = \angle QTI = a°$.

Now $\angle ABT = 90°$ $(\because AT\ is\ diameter\ of\ \odot ABC)$

$\therefore \angle PBT = 90°$

Also, $\angle PIT = 90°$

∴ PBTI is cyclic.

$\therefore \angle PBI = \angle PTI =$
$a° \ (angle \ in \ the \ same \ segment)$

$\therefore \angle IBD = \angle ABD - \angle ABI = b - a$

$\angle TBC = \angle TAC = 90° - b$

$\therefore \angle IBT = \angle IBD + \angle DBT$

$= b - a + 90° - b = 90° - a$

But since PBTI is cyclic,

$\angle IPT = \angle IBT = 90° - a$ …….(1)

$\angle BPT = 180° - \angle TPA = 180° - \angle API -$
$\angle IPT$

$= 180° - b - 90° + a$ ……..(2)

$= 90° + a - b$

But APT is a tangent to circle PQ.

$\therefore \angle BPT = \angle PQT = \angle IQT$

From (1) and (2), we get

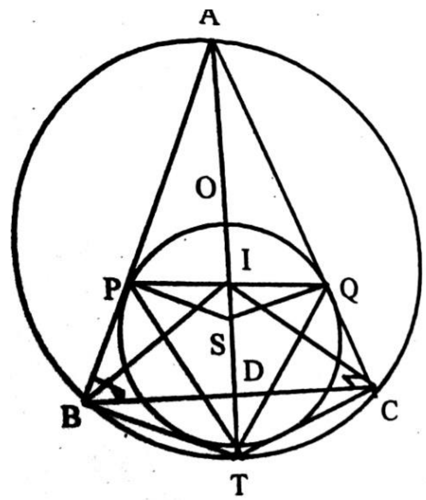
$90° + a - b = 90° - a$

$2a = b$

$\therefore \angle IBD = b - \angle PBI = 2a - a = a$

$\therefore \angle IBD = \angle PBI$

$\therefore BI$ is the angle bisector of $\angle B$



95. Area of the right angled ΔACB

$$= \frac{1}{2} AC \times BC = \frac{1}{2}(x + r)(y + r)$$

$$= \frac{1}{2}\{xy + r(x + y) + r^2\}$$

$$\Rightarrow AC \times BC = xy + rAB + r^2$$

$$\Rightarrow xy = AC \times BC - rAB - r^2$$

Now $AI^2 \times BI^2 = (x^2 + r^2)(y^2 + r^2)$

$= x^2y^2 + r^2(x^2 + y^2) + r^4$

$= x^2y^2 + r^2[(x + y)^2 - 2xy] + r^4$

$= x^2y^2 + r^2[AB^2 - 2xy] + r^4$

$= x^2y^2 + r^2AB^2 - r^2 2xy + r^4$

$= r^2AB^2 + (r^2 - xy)^2$

$= r^2AB^2 + [r^2 - AC \times BC + rAB + r^2]^2$

$= r^2AB^2 + [2r^2 - AC \times BC + rAB]^2$

Area of the $\Delta ABC = r(r + x + y)$

$= r(r + AB)$

$= r^2 + rAB = \frac{1}{2} AC.BC$

$\Rightarrow AC \times BC = 2r^2 + 2r.AB$

$\therefore AI^2 \times BI^2 = r^2AB^2 + [2r^2 - 2r^2 - 2rAB + rAB]^2$

$= r^2AB + r^2AB^2 = 2r^2.AB^2$

$\therefore AI.BI = \sqrt{2r^2.AB^2} = \sqrt{2}r.AB.$

96. AA_1, BB_1 and CC_1 are parallel line segment and hence

$$\frac{CC_1}{A_1A} = \frac{C_1B}{AB} \dots\dots (1)$$

$$Also, \qquad \frac{CC_1}{B_1B} = \frac{AC_1}{AB} \dots\dots.(2)$$

Adding (1) and (2), we have

$$\frac{CC_1}{A_1A} = \frac{CC_1}{B_1B} = \frac{C_1B + AC_1}{AB} = \frac{AB}{AB} = 1 \dots\dots (3)$$

Dividing (3) by CC_1, we get

$$\frac{1}{A_1A} + \frac{1}{B_1B} = \frac{1}{CC_1}$$

Note that ABB_1A_1 is a trapezium and C_1C_2 is the harmonic mean of the parallel sides AA_1, BB_1 and C_1C_2 is parallel to the parallel sides.

97. We can construct $22°30'$ or $22\frac{1°}{2} = \frac{90°}{4}$.
Draw an isosceles right angled triangle BCA and extend CA to D such that $AD = AB$.

$$Now, \angle ADB = \angle ABD = 22\frac{1°}{2}$$

Let us take $AC = BC = 1\ units$

Then $AB = AD = \sqrt{2}$ units

And $BD = \sqrt{1 + \left(1 + \sqrt{2}\right)^2}$ units

$= \sqrt{4 + 2\sqrt{2}}$ units

$$u = \cot 22\frac{1°}{2} = \frac{CD}{CB} = \sqrt{2} + 1$$

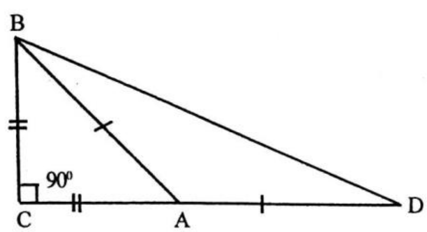
$\Rightarrow (u - 1) = \sqrt{2}$

$\Rightarrow u^2 - 2u - 1 = 0$

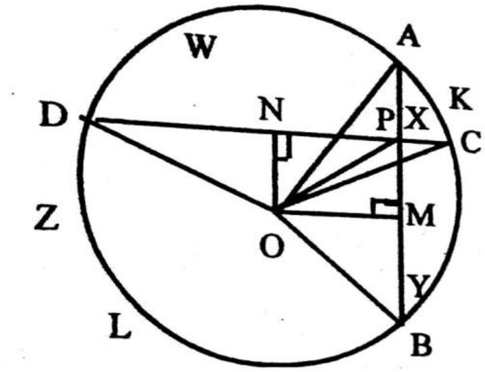
$$and\ v = \frac{1}{\sin 22\frac{1°}{2}} = \frac{BD}{BC} = \sqrt{4 + 2\sqrt{2}}$$

$\Rightarrow (v^2 - 4)^2 = 8$

$\Rightarrow v^4 - 8v^2 + 8 = 0$



98. Let P be the point of intersection of the ⊥r chord AB and CD and let the centre O, belong to the part Z. Let K and L be points on the arc AC and BD respectively as shown in the figure. Let M and N be the midpoints of the chord AB and CD respectively. Since AB and CD intersect each other at right angle,



We have $\widetilde{BOD} + \widetilde{COA} = \pi$

We have E(AOCK) + E(BODL) $= \frac{1}{2}\pi R^2$

$$E(X) + E(Z) = E(AOCK) + E(BODL) - E(OAP) - E(COP) + E(POD) + E(POB)$$

$$= \frac{1}{2}\pi R^2 - \frac{1}{2}OM \times AP - \frac{1}{2}ON \times CP + \frac{1}{2}ON \times DP + \frac{1}{2}OM \times BP$$

$$= \frac{1}{2}\pi R^2 + \frac{1}{2}OM(BP - AP) + \frac{1}{2}ON\,(DP - CP)$$

$$= \frac{1}{2}\pi R^2 + \frac{1}{2}OM[(BM + MP) - (AM - MP)] + \frac{1}{2}ON[(DN + PN) - (CN - PN)]$$

$$= \frac{1}{2}\pi R^2 + OM.MP + ON \times NP$$

$$= \frac{1}{2}\pi R^2 + 2E(ONPM)$$

Where E(ONPM) is the area of the rectangle ONPM.

Similarly,

$$E(Y) + E(W) = \pi R^2 - \left[\frac{1}{2}\pi R^2 + 2E(ONPM)\right]$$

$$= \frac{1}{2}\pi R^2 - 2E(ONPM)$$

The quantity

$$\frac{E(X) + E(Z)}{E(Y) + E(W)} = \frac{\frac{1}{2}\pi R^2 + 2E(ONPM)}{\frac{1}{2}\pi R^2 - 2E(ONPM)}$$

The maximum quantity thus corresponds to the maximum of 2E (ONPM) which is R^2 and hence the maximum quantity of

$$\frac{E(X) + E(Z)}{E(Y) + E(W)} = \frac{R^2\left(\frac{1}{2}\pi + 1\right)}{R^2\left(\frac{1}{2}\pi - 1\right)} = \frac{\pi + 2}{\pi - 2}$$

And the minimum quantity is obtained when 2E (ONPM)$= 0$ (i.e., when the perpendicular chord are two $\perp r$ diameters)

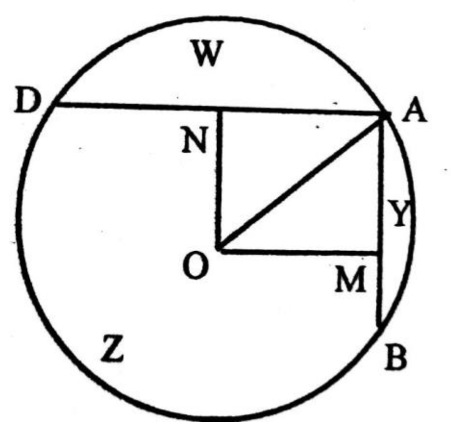
Then the minimum value of

$$\frac{E(X) + E(Z)}{E(Y) + E(W)} = 1$$

When O ∈ the region Y, then the minimum value can be shown to be $\frac{\pi-2}{\pi+2}$ and the maximum value is 1.

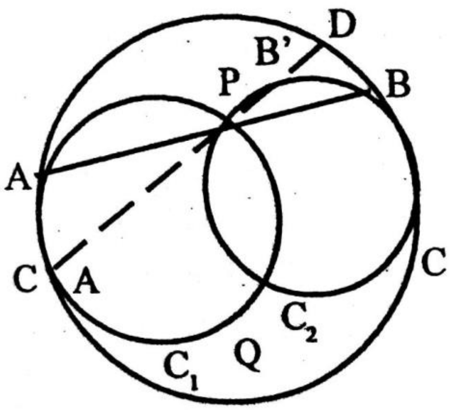
Note: The maximum is reached when we have the conditions as shown in the adjacent figure.

$$OM.ON = \frac{R}{\sqrt{2}}.\frac{R}{\sqrt{2}} = \frac{R^2}{2}$$



99. Let C_1, C_2, be two circles. We first show that if APB is a straight line such that there is a circle C touching C_1 at A and C_2 at B, then AB is the segment giving the required maximum.

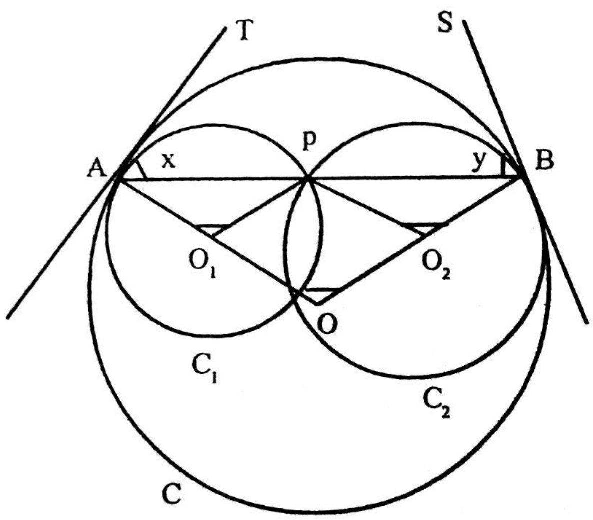
Let A'P and PB' be any other chord so that A'PB' may be collinear and the extension of these chords meet the circle C at C and D.



$CP.PD = AP.PB > A'P \times PB'$

$\therefore AP.PB$ is maximum.

Now we need to construct such a chord APB. For this we need to construct a circle C touching C_1 and C_2 at points A and B so that APB are collinear. Let us find the properties of the points A and B.



Let O be the centre of the circle C and O_1 and O_2 be the centres of the circles C_1 and C_2. Now C and C_1 touch at A.

$\therefore AO_1O$ are collinear. Similarly BO_2O are collinear. Let AT, BS be the common tangents to circles C and C_1 and C and C_2 respectively.

Let $\angle PAT = x$ and $\angle PBS = y$ since AT is tangent to circle C.

$$\angle PAT = x$$
$$= \frac{1}{2}\angle AOB \ \ (angle\ in\ the\ alternate\ segment\ theorem)$$

Since BS is tangent to circle C.

$$\angle PBS = y = \frac{1}{2}\angle AOB$$

$\therefore x = y$. Since AT is tangent to circle C_1, we get

$$\angle PAT = x = \frac{1}{2}\angle AO_1P$$

Similarly, since BS is tangent to circle C_2, we get

$$\angle PBS = y = \frac{1}{2}\angle BO_2P = x$$

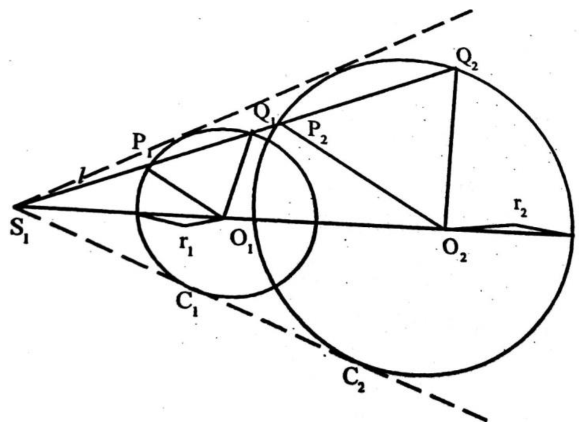
$$\therefore \angle AO_1P = \angle AOB = \angle BO_2P$$

$$\therefore \Delta AO_1P \approx \Delta PO_2B$$

$$\therefore \frac{AP}{PB} = \frac{AO_1}{PO_2} = \frac{r_1}{r_2}$$

Therefore the line segment AB must be such that P divides AB internally in the ratio $r_1 : r_2$. Further $PO_2 || OO_1$ and $PO_1 || OO_2$.

So join PO_1 and PO_2. Through O_1 draw a line parallel to PO_2 to meet circle C_1 in A. Through O_2 draw a line parallel to PO_1 to meet the circle C_2 in B. Now these two parallel lines drawn meet at O. If we draw a circle with O as centre and radius OA= OB, the circle touches C_1 at A and C_2 at B. Note, we can prove that APB are collinear and AB is the required chord.

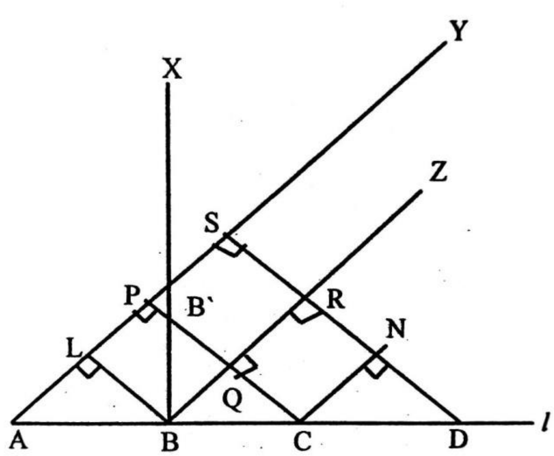


Note: In the previous problem the line AB and O_1O_2 meet in a point S_1 say. This point S_1 divides O_1O_2 externally in the ratio r_1:r_2. The point S_1 is called the external centre of similitude of the 2 circles C_1 and C_2. If we draw any line l through S_1 meeting C_1 in P_1, Q_1 and C_2 in P_2, Q_2 then P_2, Q_2 then $O_1P_1||O_2P_2$ and $O_1Q_1||O_2Q_2$.

Moreover the direct common tangent to the two circles C_1 and C_2 meet at S_1.

100. Draw BB' $\perp r$ to l and BB'=CD. Join AB' and extend it to Y. Through C and D draw perpendiculars to AY meeting AY at P and S.

Through B draw BZ perpendicular to CP and SD meeting them at Q and R respectively. PQRS is the required square.



Proof: Draw BL and CN $\perp r$ to AS and SD respectively.

$$\Delta LBB' \equiv \Delta NCD$$

$$as\ \angle LBB' = 90° - \angle ABL = \angle LAB = \angle NCD$$

$$and\ BB' = CD and\ \angle BLB' = \angle CND = 90°$$

$$PQ = LB = CN = QR = PS$$

Thus the adjacent sides of the rectangle PQRS are equal and hence it is a square.

If B' is constructed on the opposite half−plane, we get P'Q'R'S', the reflection of PQRS about the line l and lying on the opposite half plane.

In fact, this construction is possible, even if A, B, C, D are such that no three of these are collinear (i.e., for any four points in the plane).

This construction exactly follows the same procedure.

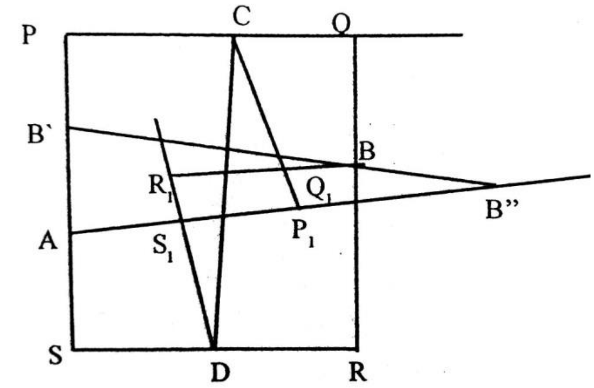
BB' (BB'') is perpendicular to CD and equal to CD. Join AB' (or AB').

Draw CP, DS $\perp r$ to AB' produced and extend PC and SD. Through B draw BR and BQ perpendicular to SR and PQ, PQRS is the required square.

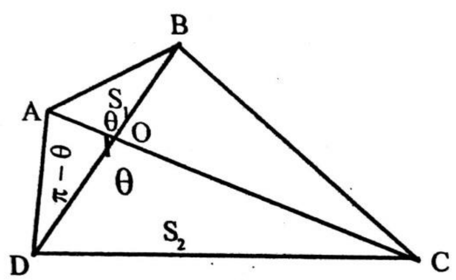
Draw CP_1, DS_1 $\perp r$ to AB'' and through B draw $\perp r$ to CP_1 and DS_1 produced meeting them at Q_1 and R_1 respectively.

$P_1Q_1R_1S_1$ is the required square.

Thus there are two solutions.



101. Let m$\angle AOB = \theta$ $area\ of\ \Delta AOB = s_1 = \frac{1}{2}OA.OB.\sin\theta$.



Similarly $s_2 = \frac{1}{2}OD.DC.sin\theta$

$$and\ s = \frac{1}{2}\{OA.OB.sin\theta + OD.OC.\sin\theta + OA.OD.\sin(\pi-\theta) + OB.OC.\sin(\pi-\theta)\}$$

$$= \frac{1}{2}\sin\theta\ [(OA.OB + OD.OC + OA.OD + OB.OC)] \ldots\ldots(1)\quad [\sin(\pi-\theta) = \sin\theta]$$

$$2\sqrt{OA.OB.OC.OD} = 2\sqrt{OA.OD.OC.OB} \leq OA.OD + OC.OB\ (applying\ AM - GM\ inequality) \ldots\ldots(2)$$

$$OA.OB + OC.OD + 2\sqrt{OA.OB.OC.OD} = \left(\sqrt{OA.OB} + \sqrt{OC.OD}\right)^2$$

But $OA.OB + OC.OD + 2\sqrt{OA.OB.OC.OD}$

$$= OA.OB + OC.OD + 2\sqrt{(OA.OD).(OB.OC)}$$

$$\leq OA.OB + OC.OD + OA.OD + OB.OC\ [by\ (1)]$$

$$So, \sqrt{OA.OB} + \sqrt{OC.OD} \leq \sqrt{OA.OB + OC.OD + OA.OD + OB.OC}\ [by\ (2)]$$

$$\sqrt{\frac{\sin\theta}{2}}\left[\sqrt{OA.OB} + \sqrt{OC.OD}\right] \leq \sqrt{\frac{\sin\theta}{2}}\sqrt{OA.OB + OC.OD + OA.OD + OB.OC}$$

$$i.e., \sqrt{s_1} + \sqrt{s_2} \leq \sqrt{s}$$

AM−GM inequality becomes an equality

When $OA.OD = OB.OC$

$$or, \quad \frac{OA}{OC} = \frac{OB}{OD}$$

That is in $\Delta s\ AOB$ and COD if $\frac{OA}{OC} = \frac{OB}{OD}$ and we know that the vertically opposite angles $\angle AOB = \angle COD$.

That if Δs AOB and COD are similar, their equality

$\sqrt{s_1} + \sqrt{s_2} = \sqrt{s}$ holds

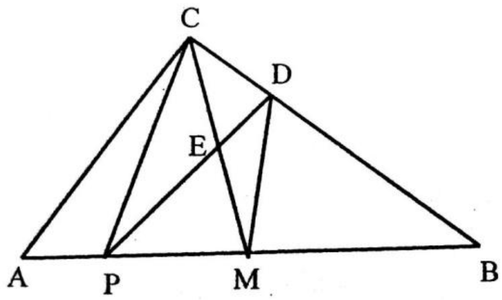
Thus, the similarity of triangles implies,

$\angle OBA = \angle ODC$, which implies that AB|| CD.

102. Join PD and MC and let them intersect at E.

Area of $\Delta BPD = Area\ of\ \Delta BMD + area\ of\ \Delta MDC$

($\Delta MDP = \Delta MDC$ as both the triangles lie on the same base MD and between the same parallels PC and MD)



$= Area\ of\ \Delta CMB$

$= \frac{1}{2} Area\ of\ \Delta ABC\ (as\ M\ is\ the\ midpoint\ of\ AB)$

$Thus, \frac{area\ of\ \Delta BPD}{area\ of\ \Delta ABC} = \frac{\frac{1}{2} Area\ of\ \Delta ABC}{Area\ of\ \Delta ABC} = \frac{1}{2}.$
$Thus, r = \frac{1}{2}\ (independent\ of\ P)$

103. Since AE is the diameter

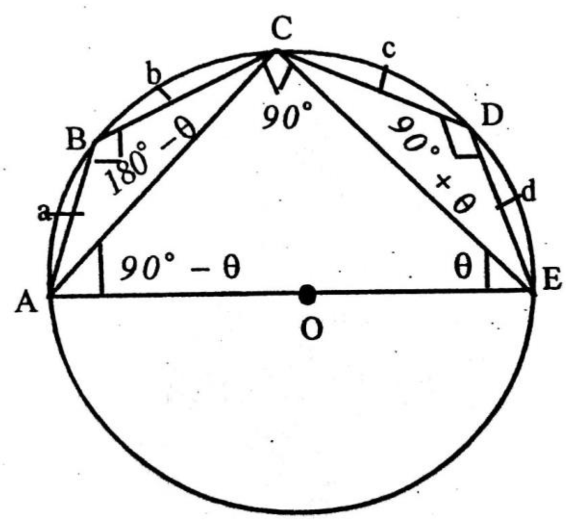
$\angle ACE = 90°$

$and\ AC^2 + CE^2 = AE^2 = 2^2 = 4$

By cosine formula (for ΔABC)

$AC^2 = a^2 + b^2 - 2ab\cos(180° - \theta)$

$= a^2 + b^2 + 2ab\cos\theta$



Similarly in $\Delta\ CED$

$CE^2 = c^2 + d^2 - 2cd\cos(90° + \theta)$

$= c^2 + d^2 + 2cd\sin\theta$

$\therefore AC^2 + CE^2 = a^2 + b^2 + c^2 + d^2 + 2ab\cos\theta + 2cd\sin\theta$

$In\ \ \Delta ACE, \frac{AC}{AE} = \sin\theta$

$\Rightarrow AC = 2\sin\theta > b\ (\because AE = 2) \ldots\ldots(1)$

$and \frac{CE}{AE} = \cos\theta (AE = 2)$

$\Rightarrow CE = 2\cos\theta > c \ldots\ldots(2)$

(Because in Δs ABC and CDE, $\angle B\ and\ \angle C$ are obtuse angles and AC is the greatest side of ΔABC and CE is the greatest side of $\Delta\ CDE$).

$\therefore\ AC^2 + CE^2 = a^2 + b^2 + c^2 + d^2 + 2ab\cos\theta + 2cd\cos\theta = 4$

$\Rightarrow a^2 + b^2 + c^2 + d^2 + ab.2\cos\theta + cd.2\sin\theta = 4$

$\Rightarrow a^2 + b^2 + c^2 + d^2 + abc + bcd < 4$ [by (1) and (2)]

104. If a is side of the rhombus, then area of the rhombus is

$\frac{1}{2}a^2\sin 2\theta \times 2$

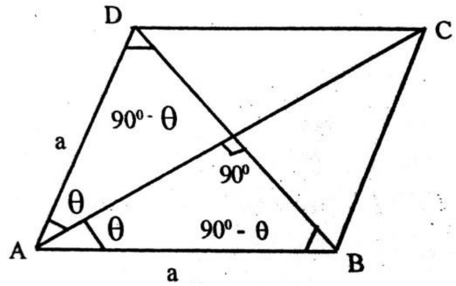
But by hypothesis, this area is equal to $\frac{1}{2}a^2$ i.e.,

$\frac{1}{2}a^2 = a^2\ sin\ 2\theta$

$\Rightarrow \sin 2\theta = \frac{1}{2}$

$\Rightarrow 2\theta = 30°\ or\ 150°$

$\Rightarrow \theta = 15°\ or\ 75°$



[If the acute angle of the rhombus is 30°, the other angle which is obtuse is 150°]

$By\ sine\ formula,$

$$\frac{BD}{\sin 2\theta} = \frac{AB}{\sin(90-\theta)}\ (in\ \Delta ABD)$$

$$\Rightarrow BD = \frac{a \times 2\sin\theta\cos\theta}{\cos\theta} = 2a\sin\theta$$

$$Again\ \frac{AC}{\sin(180-2\theta)} = \frac{a}{\sin\theta}\ (in\ \Delta ABC)$$

$$AC = \frac{a\sin 2\theta}{\sin\theta} = \frac{2a\sin\theta\cos\theta}{\sin\theta} = 2a\cos\theta$$

$$AC : BD = \cos\theta : \sin\theta$$

$$[If\ \theta = 15°, them\ AC > BD\ and\ if\ \theta = 75°, BD > AC]$$

$$AC : BD = \cos 15° : \sin 15°$$

$$= \sin 75° : \sin 15°$$

$$= \sin(45° + 30°) : \sin(45° - 30°)$$

$$= \sin 45°\cos 30° + \cos 45°\sin 30° : \sin 45°\cos 30° - \cos 45°\sin 30°$$

$$= \tfrac{1}{2}(\sqrt{3}+1) : \tfrac{1}{2}(\sqrt{3}-1)$$

$$= (\sqrt{3}+1) : (\sqrt{3}-1)$$

$$or,\quad \frac{AC}{BD} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = (2+\sqrt{3})$$

105. We should find the circumference of circle on AB as diameter.

$CD = 4\ cm$

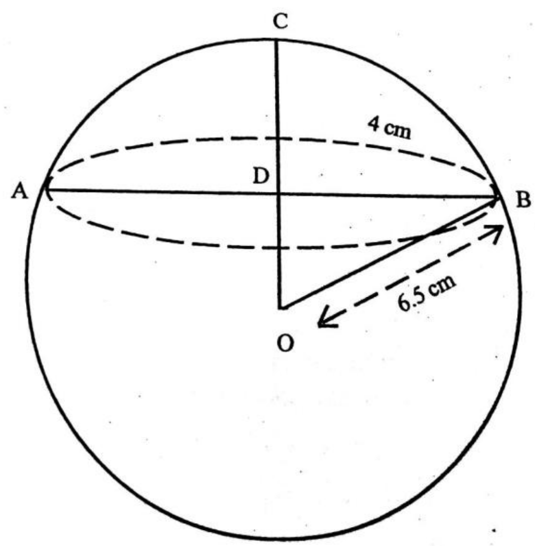
$OC = OB = \dfrac{13}{2} = 6.5\ cm$

$So, OD = 6.5\ cm - 4cm = 2.5\ cm$

$So, DB = \sqrt{(6.5)^2 - (2.5)^2} = 6\ cm$

So the circumference of the circle is

$2\pi \times 6\ cm = 12\pi\ cm$



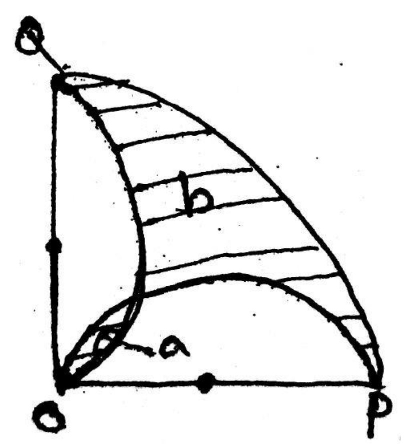
106. Area of the quadrant = areas of the two semicircles +b – a [Since sum of the areas of the two semicircles include the area shaded 'a' twice)

$$i.e., \frac{1}{4}\pi r^2 = \frac{1}{2}\pi\left(\frac{r}{2}\right)^2 + \frac{1}{2}\pi\left(\frac{r}{2}\right)^2 + b - a$$

$$\Rightarrow \frac{1}{4}\pi r^2 = \frac{1}{4}\pi r^2 + b - a$$

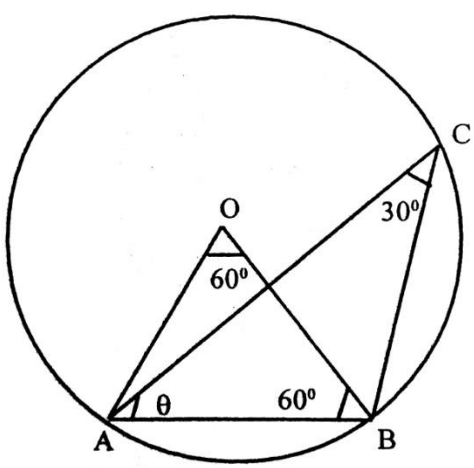
$$\Rightarrow b - a = 0$$

$$\Rightarrow a = b$$



107. To solve this problem we use the sine formula from trigonometry.

In the diagram,



$\angle ACB = \frac{1}{2}\angle AOB = \frac{60°}{2} = 30°$

If $\angle CAB = \theta \; then \; \angle ABC = (150° - \theta)$

$$By\ sine\ rule\ \frac{AB}{\sin 30°} = \frac{BC}{\sin\theta} = \frac{AC}{\sin(150° - \theta)}$$

$$\Rightarrow AC = 2 \times AB\sin(150° - \theta)$$
$$= 2\sin(150° - \theta)$$

$$and\ BC = 2AB\sin\theta = 2\sin\theta$$

$$AC^2 + BC^2 = 4sin^2\,\theta + 4sin^2(150° - \theta)$$
$$= 2[2\,sin^2\theta$$
$$+ 2sin^2(150° - \theta)]$$

$\because 2sin^2\,A = (1 - \cos 2A)$

Therefore, $2[2\,sin^2\theta + 2sin^2(150° - \theta)]$

$= 2[2 - \cos 2\theta - \cos(300° - 2\theta)]$

$= 2[2 - (\cos 2\theta + \cos(300° - 2\theta)]$

$= 2[2 - (2\cos\frac{300°}{2}.\cos(150° - 2\theta))]$

$[\because \cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}]$

$= 2[2 - 2\cos 150°.\cos(150° - 2\theta)]$

$= 2[2 + \sqrt{3}.\cos(150° - 2\theta)]$

$\leq 2(2 + \sqrt{3}) \;\because \cos(150° - 2\theta) \leq 1$

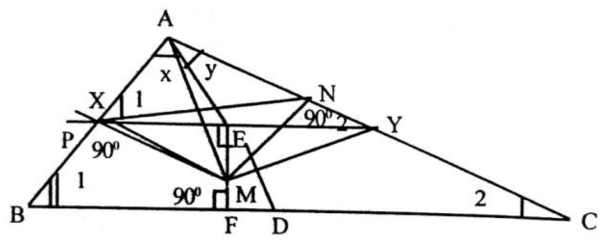
Again $AC^2 + BC^2$ is a maximum when $\cos(150° - 2\theta)$ takes the maximum value, i.e., when $150° - 2\theta \;\Rightarrow \theta = 75°$

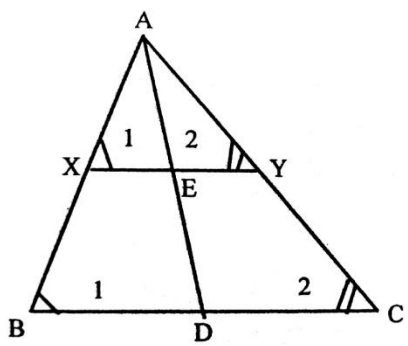
Then $\angle ABC = 75°$ therefore $\angle BAC = \theta = 75°$

Thus C takes the position at the midpoint of the major segment and AC = BC.

108. Before proving the main problem, let us prove the following:

If in ΔABC, AD is the median, XY is a line segment parallel to BC intersecting the median. AD at E, then AE is the median of ΔAXY, or in other words $XE = YE$. ΔAXE similar to ΔABD (1) and ΔAYE similar to ΔACD (2)





$$\frac{AX}{AB} = \frac{AE}{AD} = \frac{XE}{BD}; \ldots..(3)$$

$$and\,\frac{AY}{AC} = \frac{AE}{AD} = \frac{EY}{DC} \ldots \ldots (4)$$

From (3) and (4),

$$\frac{XE}{BD} = \frac{AE}{AD} = \frac{EY}{DC}$$

$$\Rightarrow \frac{XE}{BD} = \frac{EY}{DC}$$

$$\Rightarrow \frac{BD}{DC} = \frac{XE}{EY}$$

But D is the midpoint of BC and hence BD= DC

$\Rightarrow XE = EY$

$i.e., XE = YE$ …….(5)

Now, draw XY parallel to BC through E. Join AM. Join the collinear points P, E, N

MPAN is a cyclic quadrilateral as

$$\angle MPA + \angle MNA = 90° + 90° = 180°$$

Since EF is perpendicular to BC and XY is drawn parallel to BC, $\angle XEM = \angle EFB = 90°$

In the quadrilateral MPXE,

$$\angle MPX + \angle MEX = 90° + 90° = 180°$$

And hence MPXE is a cyclic quadrilateral and in the quadrilateral MENY.

$$\angle MEY = \angle MNY = 90° \ldots \ldots (6)$$

So MENY is a cyclic quadrilateral, since $\angle MEY\ and\ \angle MNY$ are subtended by MY at E and N and they are equal by (6). In Δs MEX and MEY,

$$XE = YE$$

$$\angle MEX = \angle MEY = 90°$$

ME is common

And hence $\Delta MEX\ \equiv \Delta MEY$

$$\therefore \angle MXE = \angle MYE \ldots.(7)$$

$\angle PAM = \angle PNM$ (angle on the same segment in the cyclic quadrilateral MPAN).

$= \angle ENM$

$= \angle EYM$ (angle on the same segment in quadrilateral EMYN)

$= \angle EXM$ by (7)

$= \angle EPM$ (angle on the same segment in cyclic quadrilateral MPXE)

$= \angle NPM$

$= \angle NAM$ (cyclic quadrilateral APMN)

That is AM bisects the vertical angle A of ΔABC.

That is M lies on the bisector of $\angle A$.

109. Let EF be the tangent to the circumcircle through A.

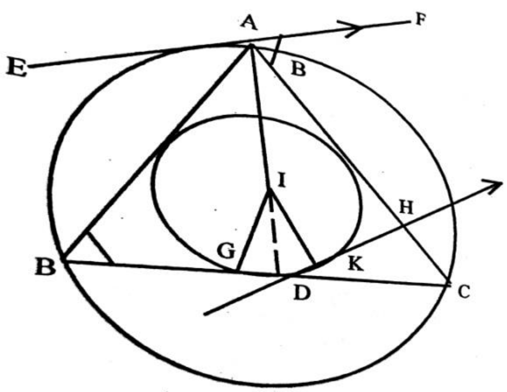
AD is the bisector of $\angle A$ and DH is parallel to EF meeting AC at H.

Let the incircle touch the side BC at G.

$\angle ADH = 180° - \angle DAF$

$= 180° - \frac{A}{2} - B$

$= C + \frac{A}{2}$



(Since $\angle HAF = \angle ABC$, being angles in alternate segments)

If the incircle touches BC at G, then

$\angle ADG = \angle DAC + \angle ACD$

(exterior angle = sum of the remote interior angles).

$= \frac{A}{2} + C$

$i.e., \angle IDG = \angle IDH \ldots.(1)$

Let the tangents through D to the incircle meet the incircle at G and K. Where G and K lie on opposite sides of ID. (Since the incircle touches the side BC at G, GD is one tangent from D, the other being DK).

$So, \angle IDG = \angle IDK$

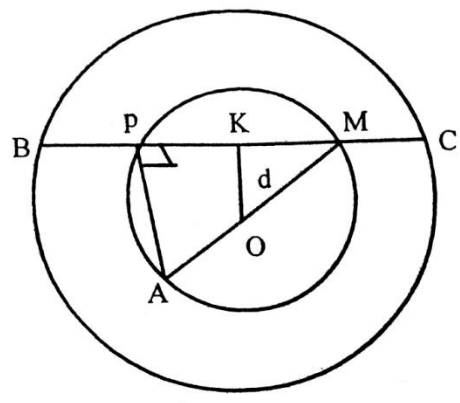
$But \quad \angle IDG = \angle IDH \ldots\ldots from\ (1)$

$Therefore \quad \angle IDK = \angle IDH,$

But both K and H are on the same side of ID and hence K is a point of DH or DH is a tangent to the incircle through D.

110. Let BC meet the smaller circle at P and M.

Through P draw PA $\perp$ r to BC meeting smaller circle at A since $\angle APM = 90°$, AM is the diameter of the smaller circle.



$or, \qquad AM = 2r$

Let OK be the $\perp$ r from O to BC.

And OK= d units; $BK = KC$;

$PK = KM$ …..(1)

Now $PA^2 + PB^2 + PC^2 = PA^2(PC - PB)^2 + 2PC.PB$

$= PA^2 + (PC - MC)^2 + 2PC.PB$ …..by (1)

$= PA^2 + PM^2 + 2PC.PB$

$= AM^2 + 2PC.PB$

$= 4r^2 + 2PC.PB$

$Now, R^2 = OB^2 = OK^2 + BK^2 = d^2 + \frac{1}{4}BC^2$

$r^2 = OM^2 = OK^2 + KM^2 = d^2 + \frac{1}{4}PM^2$

$\therefore R^2 - r^2 = \frac{1}{4}(BC^2 - PM^2)$

$= \frac{1}{4}(BC + PM)(BC - PM)$

$= \frac{1}{4}(2BK + 2PK)(2BK - 2PK)$

$= (BK + PK)(BK - PK)$

$= (CK + PK)(BP)$

$= PC.BP$

$or, 2(R^2 - r^2) = 2PC.PB$

$\therefore PA^2 + PB^2 + PC^2 = 4r^2 + 2PC.PB$

$= 4r^2 + 2(R^2 - r^2)$

$= 2R^2 + 2r^2 = 2(R^2 + r^2)$

111. Let $x = \tan a, y = \tan b, z = \tan g, \quad \frac{-\pi}{2} < a, b, g < \frac{+\pi}{2}$

$$\frac{4\sqrt{(tan^2\ a + 1)}}{\tan a} = \frac{5\sqrt{(tan^2\ b + 1)}}{\tan b} = \frac{6\sqrt{(tan^2 g + 1}}{\tan g}$$

$$\Rightarrow \frac{4}{\sin a} = \frac{5}{\sin b} = \frac{6}{\sin g} \dots .(1)$$

Again tan a tan b tan g = tan a + tan b+ tan g

$\Rightarrow \tan a(\tan b \tan g - 1) = (\tan b + \tan g)$

$\Rightarrow -\tan a = \frac{(\tan b + \tan g)}{1 - \tan b \tan g} = \tan(b + g)$

$\Rightarrow \tan(k\pi - a) = \tan(b + g)$

$\Rightarrow a + b + g = k\pi$

I taking $k = 1$, we get a + b + g= π which implies that there exists a triangle whose angles are a, b and g and whose sides opposite to these angles are proportional to 4, 5 and 6 respectively.

Let the sides of such triangle be 4k, 5k and 6k.

$s =$ semiperimeter of the triangle $= \frac{15k}{2}$

$$\tan\frac{a}{2} = \sqrt{\frac{(s - 5k)(s - 6k)}{s(s - 4k)}} = \sqrt{\frac{\frac{5k}{2} \times \frac{3k}{2}}{\frac{15}{2}k \times \frac{7}{2}k}} = \sqrt{\frac{1}{7}}$$

$$x = \tan a = \frac{2t}{1 - t^2} = \frac{2\sqrt{\frac{1}{7}}}{1 - \frac{1}{7}} = \frac{\sqrt{7}}{3}$$

Similarly, $y = \tan b = \frac{5\sqrt{7}}{9}$ and $z = \tan g = 3\sqrt{7}$

$$\left[\tan\frac{b}{2} = \sqrt{\frac{(s - 4k)(s - 6k)}{s(s - 5k)}}\ and \tan\frac{g}{2} = \sqrt{\frac{(s - 4k)(s - 5k)}{s(s - 6k)}}\right]$$

Where a, b, g are measures of the angles A, B and C of ΔABC.

112. Let f(x)= $a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x$

$f(0) = a_0 + a_1 + a_2 + a_3 = 0$ ....(1)

$f\left(\frac{\pi}{2}\right) = a_0 - a_2 = 0 \Rightarrow a_0 = a_2$.........(2)

$$f\left(\frac{\pi}{3}\right) = a_0 + \frac{1}{2}a_1 - \frac{1}{2}a_2 - a_3 = 0$$

$$\Rightarrow \frac{1}{2}a_2 + \frac{1}{2}a_1 - a_3 = 0$$

$$\Rightarrow a_3 = \frac{1}{2}(a_2 + a_1) \ldots\ldots (3)$$

$$f\left(\frac{\pi}{4}\right) = a_0 + \frac{a_1}{\sqrt{2}} - \frac{a_3}{\sqrt{2}} = 0$$

$$\Rightarrow a_2 + \frac{(a_1 - a_3)}{\sqrt{2}} = 0$$

$$or, a_2 = \frac{(a_3 - a_1)}{\sqrt{2}} \ldots\ldots (4)$$

Substituting in (1) the values obtained (2) and (3)

$$2a_2 + a_1 + \frac{1}{2}(a_1 + a_2) = 0$$

$$\Rightarrow 5a_2 + 3a_1 = 0$$

$$or, a_2 = \frac{-3}{5}a_1 \ldots\ldots (5)$$

From (4) and (5) we get

$$\left(\frac{1}{\sqrt{2}} - \frac{3}{5}\right)a_1 = \frac{1}{\sqrt{2}}a_3 \ldots\ldots (6)$$

And from (3), (5) and (6) we get

$$\left(\frac{1}{\sqrt{2}} - \frac{3}{5}\right)a_1 = \frac{1}{2\sqrt{2}}\left[\left(a_1 - \frac{3}{5}a_1\right)\right]$$

$$= \frac{1}{2\sqrt{2}} \times \frac{2}{5}a_1 = \frac{1}{\sqrt{2}.5}a_1$$

$$\Rightarrow \left(\frac{1}{\sqrt{2}} - \frac{3}{5} - \frac{1}{\sqrt{2}.5}\right)a_1 = 0$$

$$\Rightarrow \frac{\left(5 - 3\sqrt{2} - 1\right)}{5\sqrt{2}}a_1 = 0$$

$$\Rightarrow \frac{\left(4 - 3\sqrt{2}\right)}{5\sqrt{2}}a_1 = 0 \; but \; \frac{4 - 3\sqrt{2}}{5\sqrt{2}} \neq 0$$

$a_1 = 0$

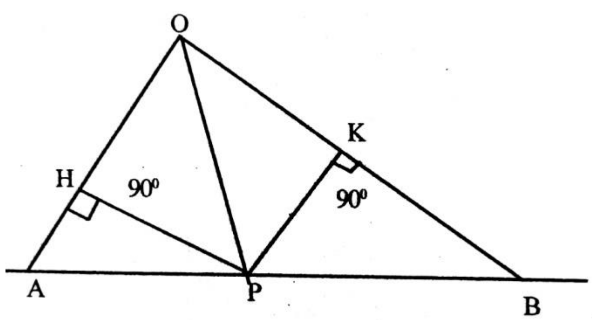
$a_3 = 0 \left[as\; a_3 = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{3}{5}\right)a_1\right]$

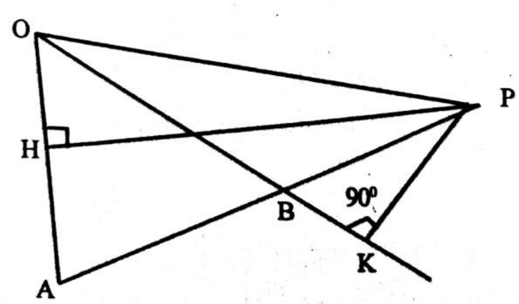
$a_2 = \frac{-3}{5}a_1 = 0$

$a_0 = a_2 = 0$

$Thus\; a_0 = a_1 = a_2 = a_3 = 0$

113.





$$\frac{AP}{PB} = \frac{\Delta AOP}{\Delta POB} = \frac{\frac{1}{2}AO.PO.\sin AOP}{\frac{1}{2}BO.PO\sin BOP}$$

$$= \frac{AO\sin AOP}{BO\sin BOP}$$

$$or, \frac{AP}{PB} = \frac{\Delta AOP}{\Delta POB} = \frac{\frac{1}{2}OA.PH}{\frac{1}{2}BO.PK}$$

$$= \frac{\frac{1}{2}OA.OP.\sin HOP}{\frac{1}{2}OA.OP\sin POK}$$

$$= \frac{OP\sin AOP}{OP\sin POB}$$

114. Let AD intersect EF at M.

Consider the ΔIMF

$\angle MFI = \angle EFC$

$= \angle EBC$

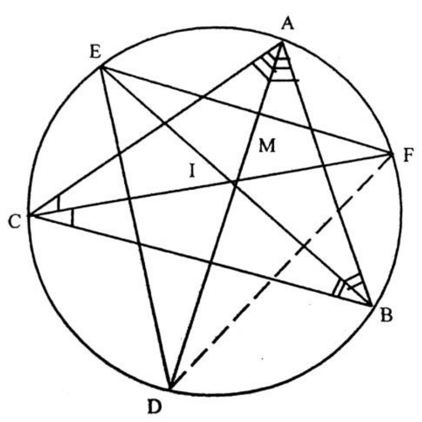
(angles in the same segment)

$= \frac{B}{2}$

$\angle MIF = 180° - \angle MIC$

$= 180° - \left[180° - \frac{A}{2} - \frac{C}{2}\right]$ $(In\ \Delta AIC)$

$= \frac{A}{2} + \frac{C}{2} = \frac{1}{2}(180° - B)$



$= 90° - \frac{B}{2}$

$\therefore \angle IMF = 180° - [\angle MFI + \angle MIF]$

$= 180° - \left(\frac{B}{2} + 90° - \frac{B}{2}\right)$

$= 90°$

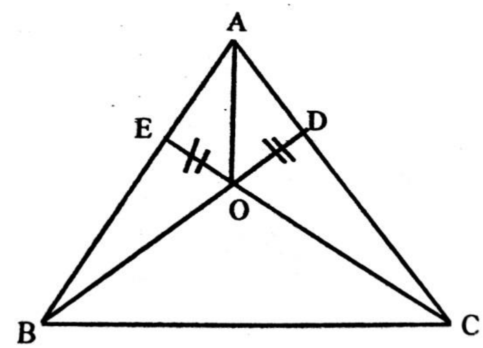
i.e., AD is perpendicular to EF.

[Similarly we can prove that BE and CF are perpendiculars to FD and ED respectively].

115. Join AO. In $\Delta AOD, m\angle OAD = \frac{A}{2}$

$m\angle ODA = m\angle BDA = C + \frac{B}{2}$

(exterior angle = sum of the remote interior angles)



$\angle AOD = 180° - \frac{A}{2} - \frac{B}{2} - C$

$= 180° - \frac{1}{2}(180° - C) - C$

$= 90° - \frac{1}{2}C$

Similarly in ΔAOE,

$m\angle OAE = \frac{A}{2}, m\angle OEA = m\angle CEA = B + \frac{C}{2}$

(exterior angle =sum of the remote interior angles)

$and\ \angle EOA = 180° - \frac{A}{2} - \frac{C}{2} - B$

$$= 180° - \frac{1}{2}(180° - B) - B$$

$$= 90° - \frac{1}{2}B$$

Using the sine formula for the two triangles ADO and AEO, we get

$$\frac{OD}{\sin\angle OAD} = \frac{AO}{\sin\angle ADO}$$

$$\Rightarrow \frac{OD}{\sin\frac{A}{2}} = \frac{AO}{\sin\left(C + \frac{B}{2}\right)}$$

$$\Rightarrow OD = \frac{AO\sin\frac{A}{2}}{\sin\left(C + \frac{B}{2}\right)} \ldots\ldots(1)$$

$$Again\ \frac{OE}{\sin\angle OAE} = \frac{AO}{\sin\angle OEA}$$

$$\Rightarrow \frac{OE}{\sin\frac{A}{2}} = \frac{AO}{\sin\left(B + \frac{C}{2}\right)}$$

$$\Rightarrow OE = \frac{AO\sin\frac{A}{2}}{\sin\left(B + \frac{C}{2}\right)} \ldots\ldots(2)$$

$$But\ OD = OE\ (given)$$

∴ From (1) and (2), we get

$$\sin\left(C + \frac{B}{2}\right) = \sin\left(B + \frac{C}{2}\right)$$

$$\Rightarrow C + \frac{B}{2} = B + \frac{C}{2} \ldots\ldots(3)$$

$$or, C + \frac{B}{2} = 180° - \left(B + \frac{C}{2}\right) \ldots\ldots(4)$$

$$\left.\begin{array}{l}\Rightarrow \frac{C}{2} = \frac{B}{2} \\ \Rightarrow \angle B = \angle C\end{array}\right\} From\ (3)$$

$$\left.\begin{array}{l}\Rightarrow \frac{3}{2}(B + C) = 180° \\ \Rightarrow B + C = 120°\end{array}\right\} from\ (4)$$

$$\therefore \angle A = 180° - 120° = 60°$$

116. Let the acute angle in the problem be a.

The harmonic mean of sin a and tan a is

$$\frac{2}{\frac{1}{\sin a} + \frac{1}{\tan a}} = \frac{2\sin a}{1 + \cos a} = \frac{4\sin\frac{a}{2}\cos\frac{a}{2}}{2cos^2\frac{a}{2}}$$

$$= 2\tan\frac{a}{2}$$

So, we should prove a < 2 tan $\frac{a}{2}$

In the figure $m\angle AOB =$ a radians and the radius of the circle with centre O is 1 unit, i.e., $OA = OB = 1$.

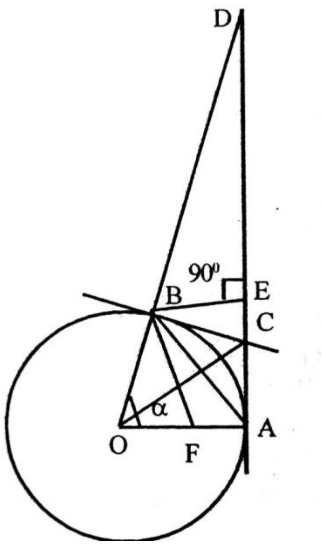
$$Arc\ AB = d < \frac{\pi}{2}$$

Let the tangents at A and B intersect at C. Let OB produced meet the tangent at A at point D and let BE ⊥r to AD.

(i) Area of the sector OAB

$$= \frac{1}{2} \times a \times 1 = \frac{a}{2}\ sq.units$$

But the sector OAB is contained in the quadrilateral OACB.



(ii) $\therefore$ Area of sector < area of the quadrilateral

$\Rightarrow$ Area of the sector < 2 area of ΔOAC [$\because \Delta OAC = \Delta OBC$]

Area of $\Delta OAC = \frac{1}{2} OA \times AC = \frac{1}{2} \times 1 \times \tan\frac{a}{2}\ sq.units$

$\therefore \frac{a}{2} < 2 < \frac{1}{2}\tan\frac{a}{2}$

$\Rightarrow a < 2\tan\frac{1}{2}\ as\ required.$

117. Draw PC'||AB and P'C' ||BC as in the figure.

$\Delta AP'C'$ is similar to Δ APC

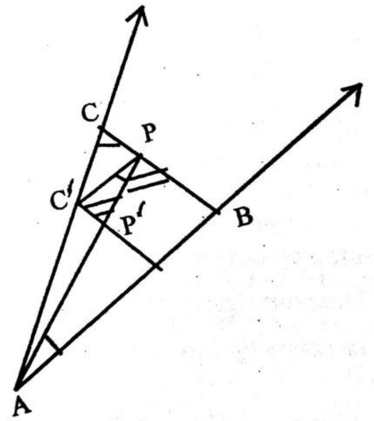
[$\because \angle P'AC' = \angle PAC, \angle ACP = \angle AC'P'$]

And $\Delta PC'P'$ is similar to ΔABP

[$\because \angle C'P'P = \angle BPA;\ \angle C'PP' = \angle BAP$]

$\therefore \frac{P'C'}{PC} = \frac{AP'}{AP} \dots\dots (1)$

$and\ PB\frac{P'C'}{PB} = \frac{P'P}{PA} \dots\dots. (2)$



Adding (1) and (2), we get

$$\frac{P'C'}{PC} = \frac{P'C'}{PB} = \frac{AP' + P'P}{PA}$$

$\Rightarrow P'C'\left(\frac{1}{PC} + \frac{1}{PB}\right) = 1\ or,\ \ \frac{1}{PB} + \frac{1}{PC} = \frac{1}{P'C'}$

If the quantity $\frac{1}{PB} + \frac{1}{PC}$ is a maximum, then P'C' should be minimum.

But C'P' is minimum if C'P' is $\perp r$ to AP. But P'C' is || to BC and P'C' $\perp r$ to AP implies BC should be perpendicular to AP. So, join the vertex A of the given angle to the given point P and draw perpendicular to AP through P, terminated by the arms of the given angle A at C and B. Now, we have got the chord BPC satisfying the hypothesis.

118. ΔABC is equilateral

$\therefore AB = BC = CA$

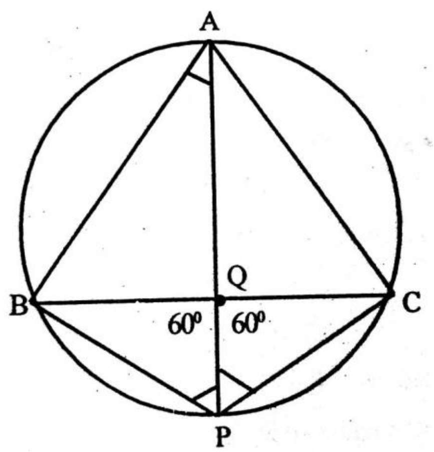
ACPB is a cyclic quadrilateral

$\therefore$ By Ptolemy's theorem, we get

$PC.AB + PB.AC = PA.BC$

Cancelling $AB = AC = BC$ we get

$PB + PC = PA$



Dividing by PB.PC, we get $\frac{1}{PB}+\frac{1}{PC}=\frac{PA}{PB.PC}$ …….(1)

$$\angle APC=\angle ABC=60°$$

$$\angle APB=\angle ACB=60°$$

In Δs ABP and CQP, we have

$\angle BAP=\angle BCP$ (angles in the same segment)

$$\angle ABP=180°-60°-\angle BAP$$

$$\angle CQP=180°-60°-\angle QCP$$

But since $\angle QCP=\angle BAP$ we get

$\angle ABP=\angle CQP$

$\therefore \Delta ABP\sim\Delta CQP$

$\therefore \frac{PA}{PC}=\frac{PB}{PQ}$

$\Rightarrow \frac{PA}{PB.PC}=\frac{1}{PQ}$

Substituting in (1), we get $\frac{1}{PB}+\frac{1}{PC}=\frac{1}{PQ}$.

119. Let a, b, c be the sides of ΔABC.

Let h_a, h_b, h_c *be the sides of* $\Delta A'B'C'$. They are also the altitudes of ΔABC.

Let h'_a, h'_b, h'_c *be the sides of* $\Delta A''B''C''$. They are also the altitudes of $\Delta A'B'C'$.

$$\frac{1}{2}ah_a=\frac{1}{2}ah_b=\frac{1}{2}ah_c=\Delta \qquad \therefore h_a=\frac{2\Delta}{a}$$

$$\frac{1}{2}h_a h'_a=\frac{1}{2}h_b.h'_b=\frac{1}{2}h_c.h'_c=\Delta$$

$$\therefore h'_a=\frac{2\Delta'}{h_a}=\frac{2\Delta}{2\Delta/a}=\frac{a\Delta'}{\Delta}$$

$$\Delta''^2=\frac{h'_a+h'_b+h'_c}{2}.\frac{h'_a+h'_b-h'_c}{2}.\frac{h'_a-h'_b+h'_c}{2}.\frac{h'_b+h'_c-h'_a}{2}$$

$$=\frac{1}{2^4}\left[\frac{a\Delta'}{\Delta}+\frac{b\Delta'}{\Delta}+\frac{c\Delta'}{\Delta}\right]\left[\frac{a\Delta'}{\Delta}+\frac{b\Delta'}{\Delta}-\frac{c\Delta'}{\Delta}\right]\left[\frac{a\Delta'}{\Delta}-\frac{b\Delta'}{\Delta}+\frac{c\Delta'}{\Delta}\right]\left[\frac{b\Delta'}{\Delta}+\frac{c\Delta'}{\Delta}-\frac{a\Delta'}{\Delta}\right]$$

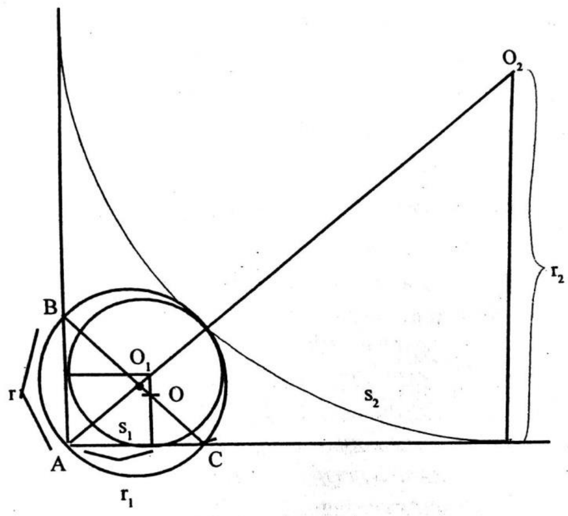
$$=\frac{\Delta^{r4}}{2^4\Delta^4}(a+b+c)(a+b-c)(a-b+c)(b+c-a)$$

$$=\frac{\Delta^{r4}}{\Delta^4}.\Delta^2=\frac{\Delta^{r4}}{\Delta^2}$$

$\Delta'=30, \Delta''=20$

$\therefore \Delta^2=\frac{\Delta^{r4}}{\Delta''^2}=\frac{30^4}{20^4}=\frac{3^4\times10^4}{2^4\times10^2}$; $\Delta=\frac{3^2\times10}{2}=45.$

120. Since both S_1 and S_2 touch AB, AC, their centres O_1, O_2 lie on the angle bisector of $\angle A$.



$In\ \Delta AOO_1$

$(O_1O)^2 = (R - r_1)^2 = R^2 + (\sqrt{2}r_1)^2 - 2R\sqrt{2}r_1\cos(B - 45°)$

[Since S and S_1 touch each other, the distance between their centres = difference in the radii $= R - r_1$.

Also $AO_1 = \sqrt{2}\, r_1$

$\angle OAO_1 = \angle OAB - \angle O_1AB\ = B - 45°$

O is the midpoint of the hypotenuse BC

$\therefore OB = OA = OC, hence\ \angle OAB = \angle OBA = \angle B$]

$$\therefore R^2 + r_1^2 - 2R_1 = R^2 + 2r_1^2 - 2R\sqrt{2}r_1\left(\frac{\cos B}{\sqrt{2}} + \frac{\sin B}{\sqrt{2}}\right) - 2Rr_1$$

$= r_1^2 - r_1(2R\sin C + 2R\sin B)$

$[\because it\ is\ a\ right\ angled\ traingle, \angle C = 90 - \angle B]$

$= r_1^2 - r_1(b + c)$

But $2R = a$

$-ar_1 = r_1^2 - r_1(b + c)$

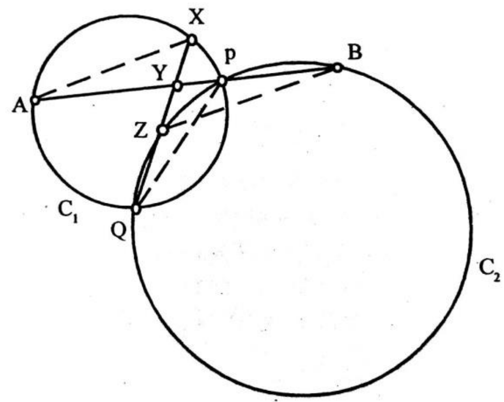
$(b + c - a) = r_1$

Similarly, from ΔO_2OA, we get cancelling r_1

$r_2 = (b + c + a)$

$\therefore r_1r_2 = (b + c)^2 - a^2 = 2bc = 4.\dfrac{1}{2}bc = 4\Delta$

121. Join PQ, BZ and AX.



In circle C_2, we have $\angle ZBP = \angle ZQP$; in circle C_1, we have $\angle PQX = \angle PAX$. Thus, we obtain $\angle ZBA = \angle BAX$. (So BZ is parallel to AX). The triangles AXY and BZY are then congruent, because by hypothesis $AY = YB$ and angles AYX and YAX are respectively equal to BYZ and YBZ. This congruence gives us $XY = ZY$, which is what we want.

122. We shall show that the locus of all such points is the union of the circumcircle and the orthocenter of the triangle ABC.

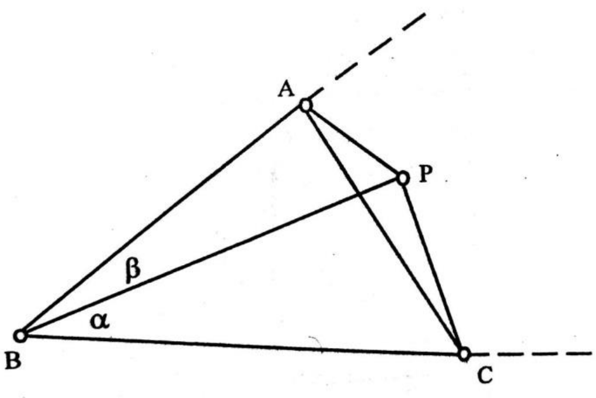
Let P be any point in the cone determined by two sides, say, BA and BC. Using the sine rule in the triangles PAC and PBC, we get

$$\angle CAP = \alpha\ or\ 180° - \alpha$$

Similarly, using the triangles CAP and BAP, we also get

$$\angle ACP = \beta \text{ or } 180° - \beta$$

Consider the case $\angle CAP = \alpha$ and $\angle ACP = 180° - \beta$



Here we get,

$\angle APC = 180° - (a + 180° - \beta) = \beta - \alpha$

Again the triangle BPC and BPA give

$\angle BAP = \angle BCP \text{ or } \angle BAP = 180° - \angle BCP.$

If $\angle BAP = \angle BCP = \gamma$, then the sum of the angles of the quadrilateral is equal to $2\beta + 2\gamma$. This implies that $\beta + \gamma = 180°$. Since $\beta \text{ and } \gamma$ are angles of a triangle, this is impossible.

If $\angle BAP = 180° - \angle BCP = 180° - \gamma$, then we get $-2\beta + 360° = 180°$. Hence $\beta = 90°$. This forces that $\angle PCA = 90°$ and AP is a diameter of the circle through A, B, C and P, is on the circumcircle of ABC. Similarly, we can dispose off the case $\angle CAP = 180° - \alpha, \angle ACP = \beta$. Finally consider the case, $\angle CAP = 180° - \alpha, and\ \angle ACP = 180° - \beta$.

Considering the triangle ACP, we see that

$$\angle APC = 180° - \angle ABC$$

Similarly, the case $\angle CAP = \alpha, \angle ACP = \beta\ gives\ that\ \angle APC\ and\ \angle ABC$ are supplementary angles. Thus, A, B, C and P are concyclic.

On the other hand, suppose P is in the cone determined by the lines, say CB and AB extended. Since

$$\angle PBC + \angle PAC = \angle PBA + \angle PCA = 180°$$

It follows that $\angle ABC\ and\ \angle APC$ are supplementary angles. Thus, triangles ABC and APC, and hence triangles ABC and BPC, have the same circumradii. Now sine rule gives

$$\angle CPB = \beta \text{ or } 180° - \beta, \angle APB = \gamma \text{ or } 180° - \gamma$$

Also, if $\angle BAP = \alpha, then\ \angle BCP = \alpha \text{ or } 180° - \alpha$

Consider the case

$$\angle CPB = \beta, \angle APB = 180° - \gamma \text{ and } \angle BCP = \alpha$$

Then

$$\angle APC = \beta + 180 - \gamma, \angle PAC + \angle PCA = \beta + \gamma + 2\alpha$$

And hence $\beta + \gamma + 2\alpha = \gamma - \beta \text{ or } \alpha + \beta = 0$ which is impossible.

If $\angle BCP = 180° - \alpha$, then we have

$$\angle APC = \beta + 180 - \gamma, \angle PAC + \angle PCA = \beta + \gamma + 180.$$

Then we would have,

$$\gamma - \beta = \beta + \gamma + 180$$

Which is impossible. Similarly we can dispose off the cases

$$\angle CPB = 180° - \beta, \angle APB = \gamma, \angle BCP = \alpha \text{ or } 180° - \alpha$$

Finally if

$$\angle CPB = \beta, \angle APB = \gamma, \angle BCP = 180^\circ - \alpha$$

Then again we get

$$\angle APC = \beta + \gamma, \angle PAC + \angle PCA = 180^\circ + \beta + \gamma$$

This forces $2(\beta + \gamma) = 0$ which is impossible. We conclude that the only possibility is

$$\angle APB = \gamma, \angle CPB = \beta \text{ and } \angle BCP = \alpha$$

In this case, we get

$$\angle APC = \beta + \gamma, \angle PAC + \angle PCA = 2\alpha + \beta + \gamma$$

This gives us

$$a = 90^\circ - (\beta + \gamma)$$

Thus, $\alpha + \beta = 90^\circ - \gamma, and\ \alpha + \beta = 90^\circ - \beta$. These imply that AP is perpendicular to CB and CP is perpendicular to AB. Hence P is the orthocenter. Similarly we can consider other regions determined by BA and CA or BC and AC.

Finally if P is a point inside the triangle, we can show that P is the orthocenter of the triangle ABC in the similar way.

Thus if P is any point satisfying the hypothesis, then either P is the orthocenter of the triangle ABC or P must be on the circumcircle of the triangle ABC.

Aliter:

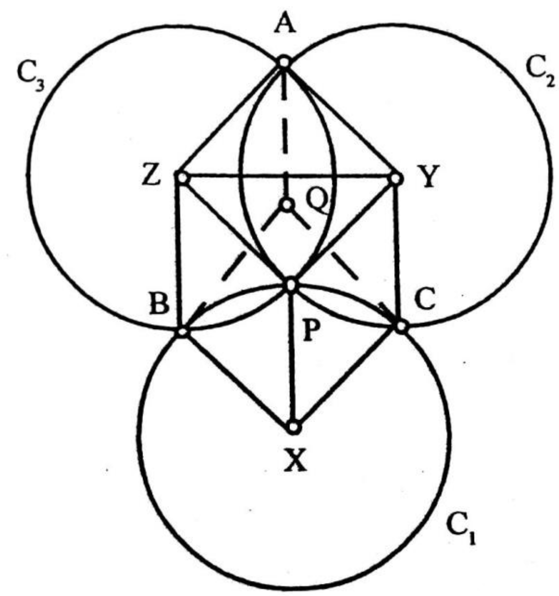
We need to know the following facts about three equal circles intersecting in a common point. If three congruent (that is, equal) circle C_1, C_2, C_3 have a common point P and A, B, C are the other three points of intersections, then

(a) The circumcircle of triangle ABC has the same radius as the three circles; and

(b) The point P is the orthocenter of triangle ABC.

A brief proof of (a) and (b) follows:

Let X, Y, Z be the centres of the circles C_1, C_2, C_3 respectively. Complete the quadrilaterals PXBZ and PXCY, join AP and ZY. Observe that PXBZ and PXCY are rhombuses and so ZB is parallel and equal to YC. Hence so are BC and ZY. Since AP is perpendicular to ZY, AP is perpendicular to BC. Similarly BP and CP are perpendicular to CA and AB respectively. Hence P is the orthocenter of triangle ABC. This proves (b).

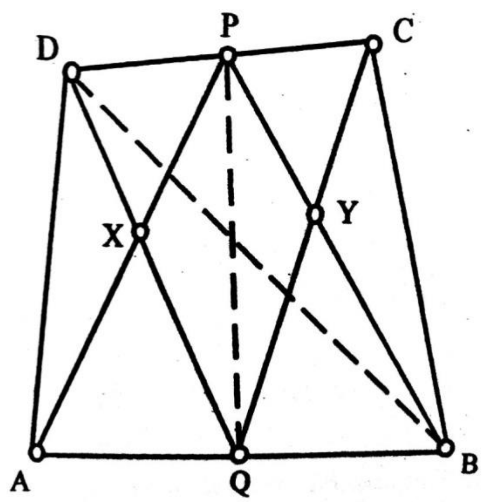


To prove (a), complete the parallelogram AYCQ, which is in fact a rhombus. So $AQ = CQ$. It is easily see that AZBQ is also a rhombus. So $AQ = BQ$. Thus Q is the circumcircle of triangle ABC and its radius ($= AQ = CY$) is the same as that of each of the three circles.

Note that we have a configuration of three equal circle such that P falls outside triangle ABC, but statements (a) and (b) are still true.

Coming to the problem : Let (XYZ) denote the circle through any three non collinear points X, Y, Z. It is given that three equal circles pass through P. Hence by (a) above, the four circles (PAB), (PBC), (PCA) and (ABC) are congruent to one another. Observe that either the three circles (PAB), (PBC), (PCA) coincide [and hence coincide with (ABC)] or the three circles are all distinct passing through the point P. Thus either P is on the circumcircle of ABC or P is the orthocenter of ABC.

123. We denote areas of triangles ABC, quadrilaterals ABCD, etc. by [ABC], [ABCD] etc. Join PQ and draw one of the diagonals, say BD. We use the fact that the median of a triangle bisects its area.



From triangles DAB (with median DQ) and BCD (with median BP), we have

$$[ADQ] = [BDQ] and\ [BPC] = [BPD]$$

Adding, we have

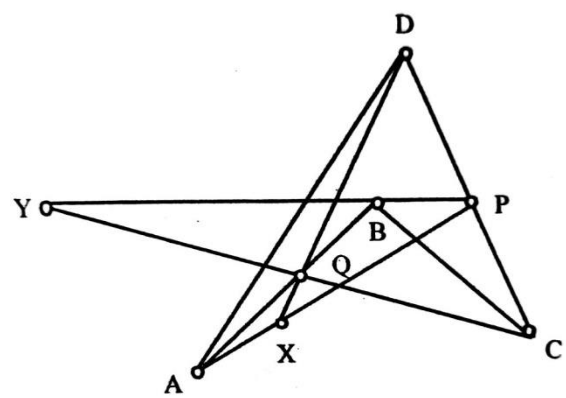
$$\begin{aligned}[ADQ] + [BPC] &= [BDQ] + [BPD] = [BPDQ] \\ &= [BPQ] + [DPQ] \\ &= [APQ] + [CPQ]\end{aligned}$$

Since PQ is a median of both the triangles APB and CQD. Writing in terms of smaller areas, we have

[AXQ] + [AXD] + [BYC] + [PYC] = [AXQ] + [PXQ] + [CPY] + [QPY].

On cancellation, this yields, [ADX] + [BCY] = [PXCY].

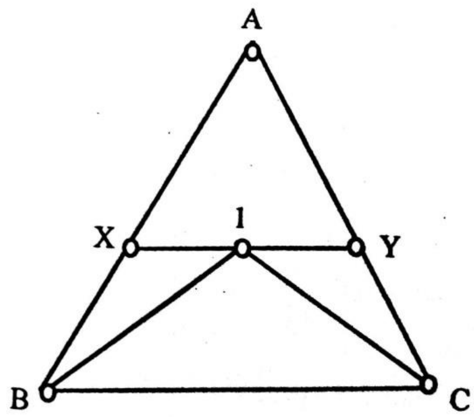
If ABCD is a concave quadrilateral and the points P, Q, X, Y are located as in the problem (see figure) then by a similar argument, we arrive at the relation | [ADX] – [BCY]|= $[PXCY]$, where the left hand side denotes the modulus of the difference of areas. The proof is left to the reader.



124. From the relation $BI^2 = BX.BA$ we see that BI is a tangent to the circle passing through A, X, I at I. Hence

$$\angle BIX = \angle BAI = \frac{A}{2} \dots\dots(1)$$

[Alternatively, one observes that in triangles BIX and BAI, $\angle IBX$ is common and BI|BX= $BA|BI$. Consequently the two triangles are similar, implying (1)].



Similarly, from the relation $CI^2 = CY.CA$. CA we obtain

$$\angle CIY = \angle CAI = \frac{A}{2} \qquad (2)$$

It is known that

$$\angle BIC = 90° + \frac{A}{2} \qquad (3)$$

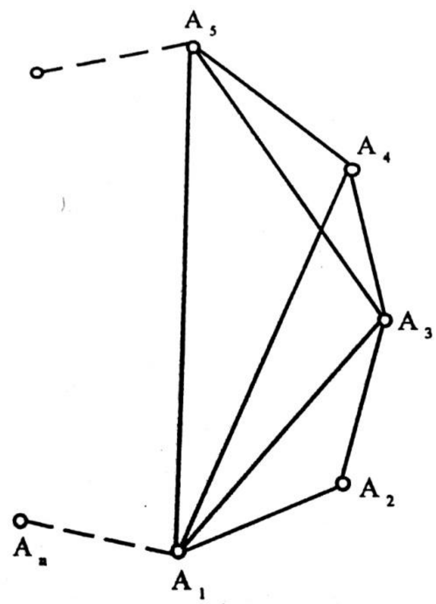
From (1), (2), (3) and the fact that X, I, Y are collinear, we obtain

$$\frac{A}{2} + \frac{A}{2} + \left(90° + \frac{A}{2}\right) = 180°$$

Solving we get $A = 60°$.

125. From the given relation, we have

$A_1A_2 . A_1A_3 + A_1A_2 . A_1A_4 = A_1A_3 . \ A_1A_4$ (1)



Also in the cyclic quadrilateral $A_1A_3A_4A_5$, we have, by Ptolemy's theorem,

$$A_4A_5.A_1A_3 + A_3A_4.A_1A_5 = A_3A_5.A_1A_4 \quad (2)$$

Since $A_1A_2....A_n$ is a regular polygon, we have

$$A_1A_2 = A_4A_5, A_1A_2 = A_3A_4, A_1A_3 = A_3A_5$$

Comparing (1) and (2), we have

$$A_1A_4 = A_1A_5$$

Since that two diagonals A_1A_4 and A_1A_5 are equal, it follows that there must be the same number of vertices between A_1 and A_4 as between A_1 and A_5. That is the polygon must be 7–sided, that is $n = 7$.

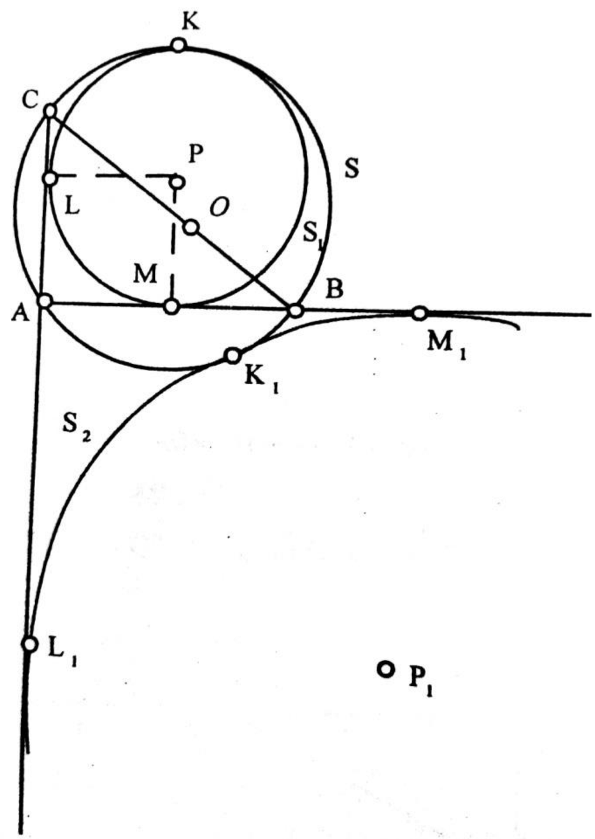
Aliter:

If O is the centre of the circle in which $A_1A_2....A_n$ is inscribed and θ is the angle which each side of the polygon subtends at O then using the relation

$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$$

Obtain as equation in θ. Solve the equation to get $\theta = \frac{2\pi}{7}$. This means $n = 7$.

126. Let S_1 touch the circle S at K, the rays AB and AC at M and L respectively. We have $PL = PM = PK = r_1$ (as P is the centre of S_1) and $R = OK = OP + r_1$. Where R is the circumradius of triangle ABC (Note that O, the midpoint of the hypotenuse BC is the circumcentre of triangle ABC). From the figure, it is clear that AMPL is a square with side r_1.



So

$BM = AB - AM = c - r_1$; and $LC = AC - AL = b - r_1$

Therefore from triangle BMP and CLP, we have

$$PB^2 = PM^2 + MB^2 = r_1^2 + (c - r_1)^2$$

$$PC^2 = PL^2 + LC^2 = r_1^2 + (b - r_1)^2$$

And

Applying Appolonius theorea to DPBC, in which PO is a median we get

$$PB^2 = PC^2 = 2(PO^2 + CO^2)$$

That is,

$$\begin{aligned} r_1^2 + (c - r_1)^2 + r_1^2 + (b - r_1)^2 \\ = 2[(R - r_1)^2 + R^2] \end{aligned}$$

Using the fact that $R = \frac{a}{2}\ and\ a^2 = b^2 + c^2$, if we solve the above equation for r_1, we obtain $r_1 = b + c - a$.

Similarly, working with S_2 we obtain $r_2 = b + c + a$.

Hence

$$r_1 r_2 = (b + c - a)(b + c + a)$$

$$= (b + c)^2 - a^2 = b^2 + c^2 + 2bc - a^2$$

$$= 2bc = 4\left(\frac{1}{2}bc\right) = 4[ABC]$$

Aliter :

Choose A as origin, AB and AC as the x–axis and y–axis respectively. Let B= $(b, 0) and\ C = (0, c)$. Then the circumcenter of triangle ABC which is at the midpoint of BC is given by $O = \left(\frac{b}{2}, \frac{c}{2}\right)$.

Any circle G which touches the positive x–axis and positive y–axis will have its center at (r, r) where r is the radius of the circle. Now the equation to the circumcircle S of the triangle ABC is

$$\left(x - \frac{b}{2}\right)^2 + \left(x - \frac{c}{2}\right)^2 = \left(\frac{a}{2}\right)^2$$

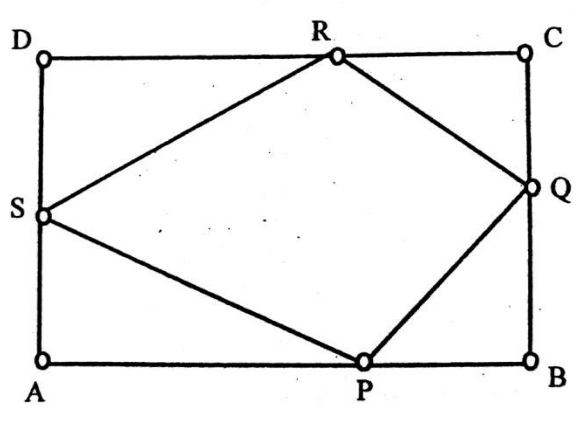
The equation to G is $(x-r)^2+(y-r)^2=r^2$. If the two circles S and G touch each other either internally (giving $G=S_2$), then we have

$$\left(r\pm\frac{a}{2}\right)^2=\left(r-\frac{b}{2}\right)^2+\left(r-\frac{c}{2}\right)^2$$

Giving $r=b+c\pm a$. Here b + c – a is the radius of the circle S_1, namely, r_1 and b + c + a is that of S_2, namely r_2.

Hence $r_1r_2=(b+c-a)(b+c+a)=4$ (area ABC), as before.

127. We have (see figure) PQ. QR > BQ. QC, QR. RS> CR. RD, etc.

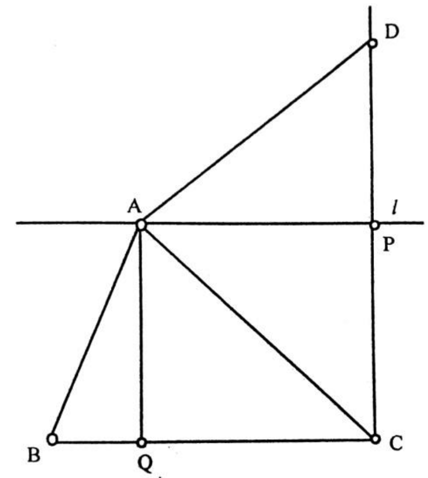


Therefore,

$$\begin{aligned}(PQ+QR+RS+SP)^2 &= PQ^2+\cdots+2PQ.QR+\cdots\\ &> (PB^2+BQ^2)+\cdots\\ &\quad+2BQ.QC+\cdots\\ &=(PA+PB)^2+(BQ+QC)^2\\ &\quad+(CR+RD)^2+(DS+SA)^2\\ &=AB^2+BC^2+CD^2+DA^2\\ &=AC^2+BD^2=2AC^2\end{aligned}$$

Hence PQ +QR + RS + SP > $\sqrt{2}$ AC.

128. Draw a line l parallel to BC through A and reflect AC in this line to get AD. Let CD intersect l in P. Join BD.



Observe that $CP=PD=AQ=h_a, AQ$ being the altitude through A. We have

$b+c=AC+AB=AD+AB\geq BD=\sqrt{CD^2+CB^2}=\sqrt{4h_a^2+a^2}$.

Which yields the result. Equality occurs if and only if B, A, D are collinear, i.e., if and only if $AD=AB$ (as AP is parallel to BC and bisects DC) and this is equivalent to $AC=BC$.

Alternatively, the given inequality is equivalent to

$$(b+c)^2-a^2\geq 4h_a^2=\frac{16\Delta^2}{a^2}$$

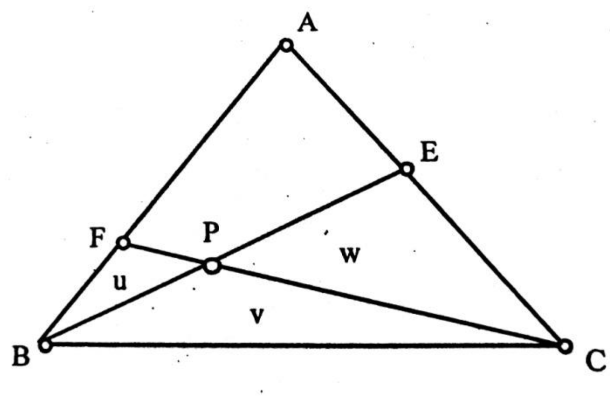
Where Δ is the area of the triangle ABC. Using the identity

$$16\Delta^2=[(b+c)^2-a^2][a^2-(b-c)^2]$$

We see that the inequality to be proved is $a^2-(b-c)^2\leq a^2$ (here we use a < b + c) which is true. Observe that equality holds if and only if $b=c$.

129.

More generally, let $[BPF] = u, [BFC] = v\ and\ [CPE] = w. Join\ AP. Let\ [AFP] = x\ and\ [AEP] = y.$



Using the triangles AFC and BFC, we get

$$\frac{x}{y+w} = \frac{FP}{PC} = \frac{u}{v}$$

This gives the equation

$$vx - uy = uw$$

Again using the triangles AEB and CEB we get another equation

$$wx - vy = -uw$$

Solving these equations, we obtain

$$x = \frac{uw(u+v)}{v^2 - uw}, y = \frac{uw(w+v)}{v^2 - uw}$$

Hence we obtain

$$x + y = \frac{uw(u+2v+w)}{v^2 - uw}$$

Putting the values $u = 4, v = 8, w = 1, we\ get\ [AFPE] = 143.$

130. Let $AB = a, BC = b, CD = c, DA = d.$

We are given that $abcd \geq 4$. Using Ptolemy's theorem and the fact that each diagonal cannot exceed the diameter of the circle we get $ac + bd = AC.BD \leq 4$. But on application of AM−GM inequality gives

$$ac + bd \geq 2\sqrt{abcd} \geq 2\sqrt{4} = 4$$

We conclude that $ac + bd = 4$. This forces $AC.BD = 4\ giving\ AC = BD = 2$. Each of AC and BD is thus a diameter. This implies that ABCD is a rectangle. Note that

$$(ac - bd)^2 = (ac + bd)^2 - 4abcd \leq 16 - 16 = 0$$

And hence $ac = bd = 2$. Thus we get $a = c = \sqrt{ac} = \sqrt{2}\ and\ similarly\ b = d = \sqrt{2}$. It now follows that ABCD is a square.

131. Let $A_1 = \{(a,b) | a, b \in \{1,2,3,\ldots,10\}, |a-b| = \{i\}, i = 0,1,2,3,4,5$

$$A_0 = \{(i,i) | i = 1,2,3,\ldots,10\} and\ |A_0| = 10$$

$$A_1 = \{(i, i+1) | i = 1,2,3,\ldots,9\} \cup \{(i, i-1) | i = 2,3,\ldots,10\}\ and\ |A_1| = 9 + 9 = 18$$

$$A_2 = \{(i, i+2) | i = 1,2,3,\ldots,8\} \cup \{(i, i-2) | i = 3,4,\ldots,10\} and\ |A_2| = 8 + 8 = 16$$

$$A_3 = \{(i, i+3) | i = 1,2,\ldots,7\} \cup \{(i, i-3) | i = 4,5,\ldots,10\}\ and\ |A_3| = 7 + 7 = 14$$

A_4
$= \{(i, i+4) | i = 1, 2, \dots, 6\}$
$\cup \{(i, i-4) | i = 5, 6, \dots, 10\}$ *and* $|A_4| = 6 + 6$
$= 12$

$A_5 = \{(i, i+5) | i$
$= 1, 2, \dots, 5\}$
$\cup \{(i, i-5) | i = 6, 7, \dots, 10\}$ *and* $|A_5| = 5 + 5$
$= 10$

$\therefore$ *The required set of pairs* $(a, b) =$ $\bigcup_{i=0}^{5} A_i$ *and the number of such pairs, (which are disjoint)* $=$ $\left|\bigcup_{i=0}^{5} A_i\right| = \sum_{i=0}^{5} |A_i| = 10 + 18 + 16 + 14 + 12 + 10 = 80$

132. From (i)

$$5, 8 \in S \dots\dots (1)$$

From (ii),

$$7, 8 \in S \dots\dots (2)$$

From (ii),

$$8, 13 \in S \dots\dots (3)$$

Therefore from eqns. (1), (2) and (3), we find that

$$5, 7, 8, 13 \in S \dots\dots (4)$$

$$S \subset \{5, 7, 8, 9, 11, 13\} \qquad (Given)$$

If at all S contains any other element other those given in (4), it may be 9 or 11 or both.

But $9 \notin S$ [$\because 9 \notin S \cup \{4, 5, 11, 13\} = \{4, 5, 7, 8, 11, 13\}$]

Again $11 \notin S$, *for* $11 \notin S \cap \{3, 5, 8, 11\} = \{5, 8\}$

$$\therefore S = \{5, 7, 8, 13\}$$

If condition (i) is not given, then S is not unique as S may be {7, 8, 13} or {5, 7, 8, 13} or {5, 7, 8, 11, 13}.

Similarly, deleting any other data leads to more than one solution to S (verify).

133. Let n(S) be 100.

$\therefore n(S) \geq n(E \cup H)$
$= n(E) + n(H) - n(E \cap H)$

$\Rightarrow 100 \geq 70 + 75 - n(E \cap H)$

$\Rightarrow n(E \cap H) \geq 45$

Similarly $100 \geq n(L \cup A) = n(L) + n(A) - n(L \cap A)$

$= 80 + 85 - n(L \cap A)$

$or, \quad n(L \cap A) \geq 65$

Now, $n(S) = 100 \geq n[(E \cap H) \cup (L \cap A)] = n[(E \cap H) + n(L \cap A) - n(E \cap H \cap L \cap A)]$

$Or, 100 \geq 45 + 65 - n(E \cap H \cap L \cap A)$

$Or, n(E \cap H \cap L \cap A) \geq 110 - 100 = 10$

That is at least 10 percent of the people must have lost all the four.

134. Every (positive) integer is a multiple of 1. So, we will first see a set consisting of a_1 and other elements:

There are 11 elements other than a_1. So the set with a_1 and another element, with one other element, 2 other elements, …, and all the 11 other elements, that is, we have to choose a_1 and 0, 1, 2, …, 11 other elements $a_2, a_3, \dots, a_{11}$.

This could be done in $11_{C_0} + 11_{C_1} + 11_{C_2} + \cdots\ 11_{C_{11}} = 2^{11}$ ways.

If a set contains a_2 as the element with the least subscript, then besides a_2 the set can have a_4, a_6, a_8, a_{10}, a_{12}, or 5 other elements, none or one or more of them. This could be done in

$$n_{C_0} + n_{C_1} + n_{C_2} + n_{C_3} + n_{C_4} + n_{C_5} = 2^5\ ways$$

Similarly for having a_3 as the element with the least subscript 3, we have a_6, a_9, a_{12} to be the elements such that the subscripts (6, 9, 12) are divisible by 3.

So, the number of subsets with a_3 as one element is

$$3_{C_0} + 3_{C_1} + 3_{C_2} + 3_{C_3} = 2^3$$

For a_4, one of the elements, the number of subsets (other elements being a_8 and a_{12}, is 2^2.

For a_5 it is 2^1 (there is just an element a_{10} such that 10 is a multiple of 5).

For a_6, it is again 2^1. (as 6/12)

For a_7, a_8, a_9, a_{10}, a_{11} and a_{12}, there is just one subset, namely, the set with these elements. This is total up to 6.

So, the total number of acceptable set according to the conditions is

$2^{11} + 2^5 + 2^3 + 2^2 + 2^1 + 2^1 + 6 = 2048 + 32 + 8 + 4 + 2 + 2 + 6 = 2102.$

If there are n elements in the set $a_1, a_2, a_3, \ldots, a_n$ then there are n multiples of 1.

$$\left[\frac{n}{2}\right]\ multiples\ of\ 2$$

$$\left[\frac{n}{3}\right]\ multiples\ of\ 3$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left[\frac{n}{n}\right]\ multiples\ of\ n$$

So that the total number of such sets is given by

$$2^{n-1} + 2^{\left[\frac{n}{2}\right]^{-1}} + 2^{\left[\frac{n}{3}\right]^{-1}} + \cdots + 2^{\left[\frac{n}{n}\right]^{-1}}$$

135. Let us number the cards, for the moment. Let us accept the case where all the cards go to one of the two players, also. With just two cards, we have possibilities,

AA. AB. BA. BB. ……..(1)

Here, AA means A gets card 1 and also card 2,

AB means A gets card 1 and B gets card 2,

BA means B gets card 1 and A gets card 2,

BB means B gets card 1 and also card 2,

Thus, for two cards we have 4 possibilities. For three card

AAA, ABA, BAA, BBA, AAB, ABB, BAB, BBB ……..(2)

i.e., for three cards there are $2^3 = 8$ possibilities. Here, if the third card goes to A, then, in (1) annex A at the end, thus getting

AAA, ABA, BAA, BBA

If it goes to B then in (1) annex B at the end, which gives

AAB, ABB, BAB, BBB.

Thus, the possibilities doubled, when a new card (third card) is included.

In fact, just with one card it may either go to A or B, giving AB.

By annexing the second card, it may give

AA BA giving (1)

AA BB

Thus, every new card doubles the existing number of possibilities of distributing the cards.

Hence, the number of possibilities with n cards is 2^n but this includes the 2 distributions where one of them gets all the cards, and the other none.

So, total number of possibilities is

$2^n - 2 = 2(2^{n-1} - 1)$.

Note: We can look at the same problem in the following way. The above distribution of cards is the same as number of possible n digit numbers where only 2 digits 1 and 2 are used, and each digit must be used at least once. This is

$2^n - 2 = 2(2^{n-1} - 1)$.

Aliter: Since n cards are dealt with and each player must get at least one card, player 1 can get r cards and player 2 get (n−r) cards where $1 \leq r \leq n-1$. Now player 1 can get r cards in C (n, r) ways.

Total number of ways of dealing cards to players 1 and 2

$$= \sum_{r=1}^{n-1} C(n,r) = \sum_{r=0}^{n} C(n,r) - C(n,0) - C(n,n) = 2^n - 2$$

136. (a) S consists of single digit numbers, two digit numbers, three digit numbers and four digit numbers.

No. of single digit numbers $= 4$

No. of two digit numbers $4 \times 3 = 12$

(Since repetition is not allowed, there are four choices for ten's place and three choices for unit's place).

No. of three digit numbers $= 4 \times 3 \times 2 = 24$

No. of four digit numbers $= 4 \times 3 \times 2 \times 1 = 24$

$\therefore n(S) = 4 + 12 + 24 + 24 = 64$

Now, for the sum of these 64 numbers, sum of all the single digit numbers is 1 + 2 + 3 + 4 = 10.

(Since there are exactly 4 digits 1, 2, 3, 4 and their numbers are 1, 2, 3, and 4).

Now, to find the sum of all the two digit numbers:

No. of two digit numbers is 12.

The digits used in unit's place are 1, 2, 3 and 4.

In the 12 numbers, each of 1, 2, 3 and 4 occurs thrice in unit digit $\left(\frac{12}{4} = 3\right)$

Again in ten's place, each of these digits occur thrice also.

So, the sum of these 12 numbers $= 30 \times (1 + 2 + 3 + 4) + 3 \times (1 + 2 + 3 + 4) = 300 + 30 = 330$.

No. of three digits numbers is 24.

So, the number of times each of 1, 2, 3, 4 occurs in each of unit's, ten's and hundred's place is $\frac{24}{4} = 6$

So, sum of all these three digit numbers is

$$\begin{aligned} &100 \times 6(1+2+3+4)+10 \\ &\quad \times 6(1+2+3+4)+1 \\ &\quad \times 6(1+2+3+4) \\ &\quad = 6{,}000+600+60=6660 \end{aligned}$$

Similarly for the four digit numbers, the sum is computed as

$$\begin{aligned} &1000 \times 6(1+2+3+4)+100 \\ &\quad \times 6(1+2+3+4)+10 \\ &\quad \times 6(1+2+3+4)+1 \\ &\quad \times 6(1+2+3+4) \\ &\quad = 60{,}000+6{,}000+600+60 \\ &\quad = 66{,}660 \end{aligned}$$

[Since there are 24 four digit numbers, each of 1, 2, 3, 4 occurs in each of the four digits in $\frac{24}{4} =$ 6 times].

So, the sum of all the single digit, two digit, three digit and four digit numbers $= 10 + 330 + 6660 + 66660 = 73{,}660$.

(b)(i) There are just four single digit numbers 1, 2, 3 and 4.

(ii) There are $4 \times 4 = 16$ two digit numbers, as digits can be repeated.

(iii) There are $4 \times 4 \times 4 = 64$ three digit numbers.

(iv) There are $4 \times 4 \times 4 \times 4 = 256$ four digit numbers.

So, total number of numbers up to 4 digit numbers that could be formed using the digits 1, 2, 3 and 4 is 4 + 16 + 64 + 256 = 340.

Sum of the 4 single digit numbers $= 1 + 2 + 3 + 4 = 10$.

To find the sum of 16, two digit numbers each of 1, 2, 3, 4 occur in each of unit's and ten's place $= \frac{16}{4} = 4$ times.

So, the sum of all these 16 numbers is

$$\begin{aligned} &= 10 \times 4(1+2+3+4)+4(1+2+3+4) \\ &= 400+40=440 \end{aligned}$$

Similarly, the sum of all the 64 three digits numbers

$$\begin{aligned} &= 100 \times \frac{64}{4} \times (1+2+3+4)+10 \times \frac{64}{4} \\ &\quad \times (1+2+3+4)+1 \times \frac{64}{4} \\ &\quad \times (1+2+3+4) \\ &\quad = 16{,}000+1{,}600+160 \\ &\quad = 17{,}760 \end{aligned}$$

Again the sum of all the 256 four digit numbers

$$\begin{aligned} &= 1000 \times \frac{256}{4} \times (1+2+3+4)+100 \times \frac{256}{4} \\ &\quad \times (1+2+3+4)+10 \times \frac{256}{4} \\ &\quad \times (1+2+3+4)+1 \times \frac{256}{4} \\ &\quad \times (1+2+3+4) \\ &\quad = 6{,}40{,}000+64{,}000+6{,}400 \\ &\quad + 640 = 7{,}11{,}040 \end{aligned}$$

Therefore, sum of all the numbers is

$$= 10 + 440 + 17{,}760 + 7{,}11{,}040 = 7{,}29{,}250$$

137. We consider numbers like 222222 or 233200 but not 212222, since the digit 1 occurs only once.

The set of all such 6 digit can be divided into following classes.

S_1 = the set of all 6 digit numbers where a single digit is repeated 6 times.

$n(S_1) = 9$, since '0' cannot be a significant number when all its digits are zero.

Let S_2 be set of all six digit numbers, made up of three distinct digits.

Here we should have two cases S_2 (a) one with exclusion of zero as a digit and other S_2 (b) with the inclusion of zero as a digit.

S_2 (a) The number of ways, three digits could be chosen from 1, 2, …, 9 is 9_{C_3}. Each of these three digits occur twice. So, the number of six digit numbers in this case is

$$= 9_{C_3} \times \frac{6!}{2! \times 2! \times 2!} = \frac{9 \times 8 \times 7}{1 \times 2 \times 3} \times \frac{720}{8}$$
$$= 9 \times 8 \times 7 \times 15 = 7560$$

S_2 case (b) The three digit used include one zero, implying, we have to choose the other two digits from the 9 non−zero digits.

This could be done in

$$9_{C_2} = \frac{9 \times 8}{1.2} = 36$$

Since zero cannot be in the leading digit, so let us fix one of the fixed non−zero number in the extreme left. Then the other five digits are made up of 2 zeroes, 2 fixed non zero number and the another non zero number, one of which is put in the extreme left.

In this case the number of six digit numbers that could be formed is

$$\frac{5!}{2! \times 2! \times 2!} \times 2$$

(since from either of the pairs of fixed non zero numbers, one can occupy the extreme digit) $=$ 60.

So, the total number in this case $= 36 \times 60 =$ 2160.

$$\therefore n(S_2) = n(S_2 a) + n(S_2 b) = 7560 + 2160$$
$$= 9720$$

Now, let S_3 be the set of six digit numbers whose digits are made up of two distinct digits each of which occurs thrice.

Here again, there are two cases : S_3(a) excluding the digit zero and S_3 (b) including the digit zero.

S_3(a) is the set of six digit numbers, each of whose digits are made up of two non zero digits each occurring thrice.

$$\therefore n[S_3(a)] = 9_{C_2} \times \frac{6!}{3! \times 3!} = 36 \times 20 = 720$$

S_3 (b) consists of 6 digit numbers whose digits are made up of three zeroes and one of the non zero digit, occurring thrice.

If you fix one of the nine non zero digit, use that digit in the extreme left. This digit should be used thrice. So in the remaining 5 digits, this fixed non zero digit is used twice and the digit zero occurs thrice.

So, the number of 6 digit numbers formed in this case is

$$9 \times \frac{5!}{3! \times 2!} = 90.$$

$$\therefore n(S_3) = nS_3(a) + nS_3(b) = 720 + 90$$
$$= 810.$$

Now, let us take S_4, the case where the six digit number consists of exactly two digits, one of which occurs twice and the other four times.

Here again, there are two cases: S_4 (a) excluding zero and S_4(b) including zero.

If a and b are two non zero numbers a used twice and b four times, then we get

$$\frac{6!}{2! \times 4!}$$

And when a used four times, b twice, we again get

$$\frac{6!}{4! \times 2!}$$

So, when 2 of the nine non zero digit are used to form the six digit number in this case, the total numbers got is

$$9_{C_2} \times 2 \times \frac{6!}{4! \times 2!} = 36 \times 5 \times 6 = 1080$$

Thus $n[S_4(a)] = 1080$

For counting the numbers is $S_4(b)$:

In this case we may use 4 zeroes and a non zero number twice or 2 zeroes and a non zero number four times.

In the former case, assuming the one of the fixed non zero digit occupying the extreme left, we get the other five digits consisting of 4 zeroes and one non zero number.

This result in $9 \times \frac{5!}{4!\times 1!} = 45$ six digit numbers.

When we use the fixed non zero digit 4 times and use zero twice, then we get $9 \times \frac{5!}{3!\times 2!} = 90$ six digit numbers, as the fixed number occupies the extreme left and for the remaining three times it occupies 3 of the remaining digits, other digits being occupied by the two zeroes.

$$So, n(S_4) = n[S_4(a)] + n[S_4(b)]$$
$$= 1080 + 45 + 90 = 1215$$

Hence, the total number of six digit numbers satisfying the given condition

$$= n(S_1) + n(S_2) + n(S_3) + n(S_4)$$
$$= 9 + 720 + 810 + 1215$$
$$= 2754.$$

138. From the hypothesis $r \leq n - r + 1$, we get

$2r \leq n + 1.$

Each such r combination can be represented by a binary sequence b_1 b_2 b_3 …. b_n where $b_i = 1$, if i is a member of the r combination and 0, otherwise with no consecutive $b'_i s = 1$ (the above r combinations contain no consecutive integers).

The number of Is in the sequence is r.

Now, this amounts to counting such binary sequences.

Now, look at the arrangement of the following boxes; and the balls in them.

1	2	3	4	5	6	7
OO	OOO	OO	OOOO	O	O	OOO

Here, the balls stand for the binary digits zero, and the boundaries on the left and right of each box can be taken as the binary digit one. In this display of boxes and balls as interpreted gives previously how we want the binary numbers. Here, there are 7 boxes, and 6 left/right boundary for the boxes. So this is an illustration of 6 combination of non−consecutive numbers.

The reason for zeroes in the front and at the end is that we can have leading zeroes and trailing zeroes in the binary sequence $b_1, b_2, \dots, b_n$.

Now, clearly finding the r combination amounts to distribution of (n – r) balls into (r + 1) distinct boxes $[(n-r)balls\ = (n-r)zeroes$as these are r ones, in the n number sequence] such that the 2nd, 3rd … rth boxes are non empty. (The first and last boxes may or may not be empty – in the illustration of 1st and 7th may have zeroes or may not have balls as we have already had 6 combinations).

Put (r−1) balls one in each of 2nd, 3rd, …, rth boxes (so that no two 1's occur consecutively). Now we have (n – r) – (r – 1) balls to be distributed in (r + 1) distinct boxes.

139. T can be written as $T = T_1 \cup T_2, T_1 = \{(x,y,z)|x,z\ \in S, x<y\}\ and\ T_2 = \{(x,y,z)!\, x,y,z\ \in S, x+y<z\}$.

The number of elements in T_1 is the same as choosing two elements from the set S, where $n(S) = (n+1), i.e., n(T_1) = \binom{n+1}{2}$. (as every subset of two elements, the larger element will be z and the smaller will be x and y).

In T_2, we have $2\binom{n+1}{3}$ elements, after choosing 3 elements from the set S, two of the smaller elements will be x and y and they may be either taken as (x, y, z) or as (y, x, z) or in other words, every three element subset of S, say (s, b, c), the greatest is z, and the other two can be placed in two different ways in the first two positions,

$$\therefore n(T)(or\ |T|) = \binom{n+1}{2} + 2\binom{n+1}{3}$$

T can also be considered as

$$\bigcup_{i=2}^{n+1} S_i = \{(x,y,i)|x,y<i, x,y \in S\}$$

All these sets are pair wise disjoint as for different i, we get different ordered triplets (x, y, i).

Now in S_1, the first two components of (x, y, i), namely (x, y), can be any element from the set 1, 2, 3, …, i – 1.

x and y can be any member from 1, 2, 3, …, (i−1), equal or distinct.

∴ The number of ways of selecting (x, y), x, y ∈ $\{1,2,3,\ldots,(i-1)\} is\ (i-1)^2$.

Thus, $n(S_i)$ for each i is $(i-1)^2, i \geq 2$. For example,

$$n(S_2) = 1, n(S_3) = 2^2 = 4\ and\ so\ on$$

$$Now, n(T) = n\left(\bigcup_{i=2}^{n+1} S_i\right)$$

$$= \sum_{i=2}^{n+1} n(S_i)$$

(because all $S'_i s$ are pair wise disjoint)

$$= \sum_{i=2}^{n+1} (i-1)^2 = \sum_{i=1}^{n} i^2$$

And hence,

$$\binom{n+1}{2} + 2\binom{n+1}{3} = \sum_{k=1}^{n} k^2$$

140. For each positive integer k, $1 \leq k \leq 5, let\ N_k$ denote the number of permutations (p_1, p_2, …, p_6) such that $p_1 \neq 1, (p_1, p_2)$ is not a permutation of (1, 2), … $(p_1, p_2, \ldots, p_k)$ is not a permutation of (1, 2, …, k).

We are required to find N_5.

We shall start with N_1.

The total number of permutations of (1, 2, 3, 4, 5, 6) is 6! And the permutations of (2, 3, 4, 5, 6) is 5! Thus, the number of permutations in which $p = 1$is 5!.

So, the permutation in which $p \neq 1\ is\ 6! - 5! = 720 - 120 = 600$.

Now, we have to remove all the permutations with (1, 2)and (2, 1) as the first two elements to get N_2. Of these, we have already taken into account (1, 2) in considering N_1 and subtracted all the permutations starting with 1. So we should consider the permutation beginning with (2, 1).

When $p_1 = 2, p_2 = 1$ (p_3, p_4, p_5 and p_6) can be permuted in 4! ways.

So, $N_2 = N_1 - 4! = 600 - 24 = 576$.

Having removed the permutations beginning with (1, 2), we should now remove those beginning with (1, 2, 3). But, corresponding to the first two places (1, 2) and (2, 1), we have removed all the permutations. So, we should now remove the permutations with first three places (3, 2, 1), (3, 1, 2), (2, 3, 1).

Note that the first 3 positions being 1 2 3 is included in the permutations beginning with 1.

For each of these three arrangements, there are 3! Ways of arranging 4th, 5th and 6th places and hence,

$$N_3 = N_2 - 3 \times 3! = 576 - 18 = 558$$

To get N_4, we should remove all the permutations beginning with the permutations of (1, 2, 3, 4) of which the arrangement of (1, 2, 3) in the first three places have already been removed. We have to account for the rest.

So, when 4 is in the first place, 3! Arrangements of 1, 2, 3 in the 2nd, 3rd and 4th places are possible. Also, when 4 is in the second place, since we have removed the permutations when 1 occupies the first place, there are two choices for the first place with 2 or 3 and for each of these there are 2 arrangements, i.e., (2, 4, 1, 3), (2, 4, 3, 1), (3, 4, 2, 1), (3, 4, 1, 2). When 4 is in the third place, then there are first 3 arrangements (2, 3, 4, 1), (3, 2, 4, 1) and (3, 1, 4, 2).

So, the total permutations of (1, 2, 3, 4) to be removed from S_3 to get S_4 is $(6 + 4 + 3) \times 2 = 26$, because there are 2 ways of arranging the 5th and 6th places for each one of the above permutations of (1, 2, 3, 4).

$$\therefore\ S_4 = S_3 - 26 = 558 - 26 = 532$$

To get N_5; we should remove from S_4 al the permutations of (1, 2, 3, 4, 5) which have not been removed until now. When 5 occupies the first position, there are $4! = 24$ ways of getting 2nd, 3rd, 4th and 5th places which have not been removed so far.

When $p_2 = 5, p_1$ cannot be 1, so p_1 can be chosen from the other 3, viz., 2, 3 and 4, in 3 ways and 3rd, 4th and 5th places can be filled for each of the first place choice in $3 \times 2 \times 1 = 6$ ways.

So, when $p_2 = 5$, there are 18 different arrangements to be removed.

When $p_3 = 5$, the first 2 places cannot be (1, 2) so that they can be filled in (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3) and for the fourth and fifth places there are exactly two choices for each of the above arrangements for the first and second place.

So when $p_3 = 5$, the number of arrangements to be removed is $8 \times 2 = 16$.

When $p_4 = 5, p_1p_2p_3$ can be removed (241, 412, 421, 234, 243, 342, 324, 423, 432, 314, 341, 413, 431) and since there is only one choice left, we now to remove 13 arrangements when

$$p_4 = 5$$

When $p_5 = 5$, we have already removed the permutations of (1, 2, 3, 4) of the first four places to find S_4

So now $S_5 = S_4 - (24 + 18 + 16 + 13) = 534 - 71 = 463$.

So, 463 is the desired number of permutations.

141. The given set $S = \{1, 2, 3, 4, \ldots, 299, 300\}$ can be realized as the union of the three disjoint sets s_1, s_2 and s_3 with

$$s_1 = \{x \in S | x = 3n + 1, 99 \geq n \geq 0\},$$

$$s_2 = \{x \in S\ | x\ = 3n + 2, 99 \geq n \geq 0\}$$

$$and\ s_3 = \{x \in S\ | x = 3n, 100 \geq n \geq 1\}$$

Now, to get the set of all three element subsets of S, with the sum of the elements of the subset a multiple of 3, we should choose all three elements from the same set s_1, s_2 or s_3 or we should choose one element from each of the set s_1, s_2 and s_3.

We see that $n(s_1) = n(s_2) = n(s_3) = 100$.

Choosing all the three elements from either s_1, s_2 or s_3 will give $3 \times 100_{C_3}$ triplets and its sum is also divisible by 3.

Choosing the three elements, one each from s_1, s_2 and s_3 will give $100_{C_1} \times 100_{C_1} \times 100_{C_1}$ triplets and its sum is also divisible by 3.

So, the total number of 3 element subsets with the required property is

$$3 \times 100_{C_3} + 100_{C_1} \times 100_{C_1} \times 100_{C_1}$$

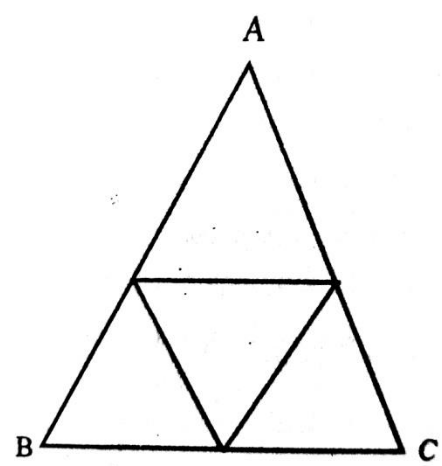
$$= \frac{3 \times 100 \times 99 \times 98}{1 \times 2 \times 3} + 100^3$$
$$= 100 \times 99 \times 49 + 1000000$$
$$= 485100 + 1000000$$
$$= 1485100$$

142. ABC is an equilateral triangle of side 1 cm. If the sides are divided into equal parts, we get 4 equilateral triangles with side ½ cm.

Again, if each of these four triangles is subjected to the above method, we get 4×4 triangles of side

$$\frac{1}{2} \times \frac{1}{2} = \frac{1}{2^2}\, cm$$

Thus, after n steps we get, 4^n triangles of side $\frac{1}{2^n}\, cm.$



Now, if we take $4^n + 1$ points inside the original equilateral ΔABC, then at least two of the points lie on the same triangle out of 4^n triangles by Pigeon Hole Principle.

Hence, the distance between them is less than or at the most equal to the length of the side of the triangle, in which they lie, that is, they are $\frac{1}{2^n} cm$ apart or they are less than $\frac{1}{2^n} cm$ apart.

143. There are $\frac{(100-1)}{3} + 1 = 34$ elements in the progression 1, 4, 7, …, 100.

Consider the following pairs:

(4, 100), (7, 97), (10, 94), …, (49, 55).

There are in all

$$\frac{49-4}{3} + 1 = 16\, pairs\, \left(or\ \frac{100-55}{3} + 1\right)$$

Now, we shall show that we can choose eighteen distinct numbers from the A.P. such that no two of them add up to 104. In the above 16 pairings of the A.P. the numbers 1 and 52 are left out.

Now, taking one of the numbers from each of the pairs, we can have 16 numbers and including 1 and 52 with these 16 numbers, we now have 18 numbers.

(Note. We can have 216 such sets of numbers, and any two of these sets will have the common elements 1 and 52).

But, no pair of numbers from these 18 numbers can sum up to 104, since just one numbers is selected from each pair, and the other number of the pair (not selected) is 104 – the number chosen.

Also $1 + 52 \neq 54$. Thus, we can choose 18 numbers, so that no two of them sum up to 104.

For getting 19 numbers (all these should be distinct), we should choose one of the 16 not chosen numbers, but then this number chosen is the 104 complement of one of the 16 numbers chosen already (among the 18 numbers).

Thus, if a set of 19 distinct elements are chosen, then we have at least one pair whose sum is 104.

144. Since $n(X) = 10$, the number of non empty, proper subsets of X is

$$2^{10} - 2 = 1022.$$

The sum of the elements of the proper subsets of X can possibly range from

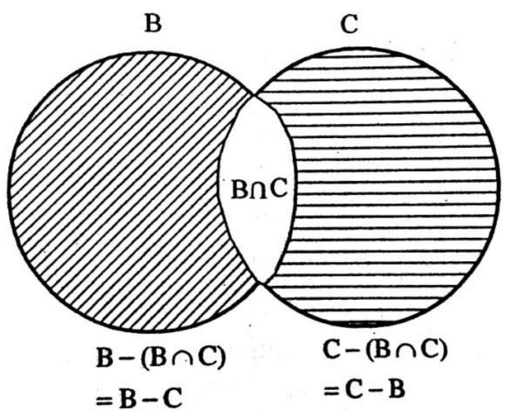
$$1\ to\ \sum_{i=1}^{9}(90 + i).$$

That is 1 to (91 + 92 + … + 99), i.e., 1 to 855.

That is, the 1022 subsets can have sums from 1 to 855.

By Pigeon−hole Principle, at least two distinct subsets B and C will have the same sum.

(Since there are 855 different sums, and so if we have more than 855 subsets, then at least two of them have the same sum).



If B and C are not disjoint, then let

$$X = B - (B \cap C)$$

$$and\ Y = C - (B \cap C)$$

Clearly, X and Y are disjoint and non empty and have the same sum of their elements Define s(A) = sum of the elements of A.

We have B and C not necessarily disjoint such that s(B)= s(C).

Now, $s(X) = s(B) - s(B \cap C)$

$s(Y) = s(C) - s(B \cap C)$

But $s(B) = s(C). Hence, s(X) = s(Y)$.

Also $X \neq \phi$. For if X is empty, then $B \subset C$ which implies s(B) < s(C) (a contradiction).

Thus, X and Y are non empty and $s(X) = s(Y)$.

145. Consider three numbered boxes whose contents are denoted as x_1, x_2, x_3 respectively. The problem now reduces to distributing 28 balls in the three boxes such that the first box has at least 3 and not more than 9 balls, the second box has at most 8 balls, and the third box has at least 7 and at most 17 balls.

At first, put 3 balls in the first box, and 7 balls in the third box. This takes care of the minimum needs of the boxes.

So, now the problem reduces to finding the number of distribution of 18 balls in 3 boxes such that the first has at most (9 – 3) = 6, the second at most 8 and the third at most $(17 - 7) = 10$.

The number of ways of distributing 18 balls in 3 boxes with no condition is

$$\binom{18+3-1}{3-1} = \binom{20}{2} = 190.$$

[The number of ways of distributing r identical objects in n distinct boxes is

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

where 'n' stands for the numbers of boxes and r for balls].

Let d_1 be the distribution where the first box gets at least 7; d_2, the distributions where the second box gets at least 9; and d_3, the distribution where the third gets at least 11.

$$|d_1| = \binom{18-7+3-1}{3-1} = \binom{13}{2} = \frac{13 \times 2}{1.2}$$
$$= 78$$

$$|d_2| = \binom{18-9+3-1}{3-1} = \binom{11}{2} = \frac{11 \times 10}{1.2}$$
$$= 55$$

$$|d_3| = \binom{18-11+3-1}{3-1} = \binom{9}{2} = \frac{9 \times 8}{1.2}$$
$$= 36$$

$$\therefore |d_1 \cap d_2| = \binom{18-7-9+3-1}{3-1} = \binom{4}{2}$$
$$= 6$$

$$|d_2 \cap d_3| = \binom{18 - 9 - 11 + 3 - 1}{3 - 1} = \binom{0}{2} = 0,$$

$$|d_3 \cap d_1| = \binom{18 - 11 - 7 + 3 - 1}{3 - 1} = \binom{2}{2} = 1$$

Therefore, $|d_1 \cap d_2 \cap d_3| = 0$

And $|d_1 \cup d_2 \cup d_3| = 78 + 55 + 36 - 6 - 0 - 1 + 0 = 162$.

So, the required number of solutions $= 190 - 162 = 28$.

[Note that, the number of ways the first box gets utmost 6, the second utmost 8 and the third utmost 10 = Total number of ways of getting 18 balls distributed in 3 boxes – (the number of ways of getting at least 7 in the first box, at least 9 in the second box and at least 1 in the third box).

And $n(A \cup B \cup C)' = n(A' \cap B' \cap C')$.

146. Let the proposer of the problems be called X, and the friends be denoted as A, B, C, D, E, F. Since X dines with all the 6 friends exactly on one day, we have the combination XABCDEF (1) for one day 1.

Thus, every five of A, B, C, D, E, F had already dined with X for a day. According to the problem, every five of them should dine on another day. It should happen in $6_{C_5} = 6$ days. The combination is XABCDE (2), XABCDF (3), XABCEF (4), XABDEF (5), XACDEF (6), XBCDEF (7).

In (1) and (2) together, X has already dined with every four friends three times, for example with ABCD, he dined on the first day the numbers above the combinations can be taken as the rank of the days X dines with his friends.

2nd and 3rd days, X dined with every three friends of them on four days, for example with ABC, 1st, 2nd, 3rd and 4th days, X has dined with every two friends, of them for five days for example with AB, 1st, 2nd, 3rd, 4th and 5th days.

With just one of them he has dined so far 6 days (with A, 1st, 2nd, 3rd, 4th, 5th and 6th days).

So, he has to dine with every one of them for one more day he should dine with XA, XB, XC, XD, XE and XF for 6 more days. Thus, the total number of days he dined so far with at least one of his friends is 1 + 6 + 6 = 13 days. In this counting, we see that he has dined with every one of them for 7 days. That shows that he has not dined with every one of them for 6 days.

But it is given that every friend was absent for 7 days. Since each one of them has been absent for 6 days already, all of them have to be absent for one more day.

Thus, he dined alone for 1 day and the total number of dinners he had is 13 + 1 = 14.

147. Now pair of the elements of S as [a, a + 2nd], [a + d, a + (2n – 1)d], …, [a + (n – 1)d, a + (n + 1)d] and one term a + nd is left out.

Now, sum of the terms in each of the pairs is 2(a + nd). Thus, each term of the pair is 2(a + nd) complement of the other term.

Now, there are n pairs. If we choose one term from each pair, we get n term. To this collection of terms include (a + nd) also.

Now, we have (n + 1) numbers. Thus, set A can be taken as the set of the above (n + 1) numbers, here no two elements of the set A add up to 2(a + nd) as no element has its 2(a +

nd) complement in A except a + nd, but then we should take two distinct elements.

If we add any more terms to A so that A contains more than (n + 1) elements, then some of the elements will now have then 2(a + nd) complement in A, so that sum of these two elements will be 2(a + nd), and hence, the result.

In the second case, we have

$$S = \{a, a + d, \dots, a + (2n + 1)d\}$$

There are 2(n + 1) elements. So, pairing them as before gives (n + 1) pairs i.e., [a, a+(2n+1)d], [a+d, a+2nd], …, [a+nd, a+(n+1)d]

Now, we can pick exactly one term from each of these (n + 1) pairs.

We get a set A of (n + 1) elements where no two of which add up to [2a + 2(n + 1)d].

Note here we need not use distinct numbers, even if the same number is added to itself, the sum will not be [2a + 2(n + 1)d]. Here again, even choosing one more term from the numbers left out, and adding it to A, A will have a pair which adds up to [2a + 2(n+1)d].

Thus, the maximum number of elements in A satisfying the given condition is (n + 1).

148. Let $S = \{1, 2, 3, \dots, 63\}$.

Let A be the set of all three elementic subsets of S such that (a + b + c) < 95, i.e.,

$$A = \{(a, b, c) | (a + b + c) < 95, (a, b, c) \in S\}$$

Similarly, let B be the set of all three elements subsets of S such that (a + b + c) > 95, where (a, b, c) ∈ S},

$$i.e., B = \{(a, b, c) | (a + b + c) > 95, (a, b, c) \in S\}$$

$$and\ C = \{(a, b, c) | (a + b + c) > 97, (a, b, c) \in S\}$$

Clearly, C is a proper subset of B because $(a, b, c) \in S, if\ (a + b + c) = 96\ then\ (a, b, c) \in B\ and\ (a, b, c) \notin C.$

However, every element of C∈ B,

As (a + b +c) > 97

$or, (a + b + c) > 95$ also

Hence, (a, b, c) ∈ $C \Rightarrow (a, b, c) \in B$ also.

Now, it is enough if we show that $n(A) = n(C)\ as\ n(C) < n(B) and\ n(A) = n(C) \Rightarrow n(A) \subset n(B).$

If (a, b, c) ∈ $A, then\ 1 \leq (a + b + c) < 95\ and\ also\ 1 \leq (a, b, c) \leq 63.$

$Therefore, 1 \leq (64 - a), (64 - b), (64 - c) \leq 63\ and\ as\ (a + b + c) < 95, (64 - a) + (64 - b) + (64 - c) = 192 - (a + b + c) > 192 - 95 = 97.$

Thus to each element of A, there is a unique element in C. Conversely, if (a, b, c) ∈ C,

$$then\ (64 - a), (64 - b), (64 - c) \in A\ for\ (64 - a) + (64 - b) + (64 - c) = 192 - (a + b + c), and\ since\ (a, b, c) \in C, (a + b + c) > 97$$

$\therefore 192 - (a + b + c) < 192 - 97 = 95.$

And thus $\{(64 - a), (64 - b), (64 - c)\} \in A$, which shows that every element of C there corresponds a unique element in A. Thus, there is a 1−1 correspondence between the sets A and C.

$$\therefore n(A) = n(C) < n(B).$$

149. Here we are using the property of tangent functions of trigonometry.

Given a real number a, we can find a unique real number A, lying between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ i.e., lying in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Such that $\tan A = a$, as the tangent function in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is continuous and strictly increasing and covers R completely.

Therefore, corresponding to the five given real numbers

$a_i (i = 1, 2, 3, 4, 5)$, we can find five distinct real numbers

$A_i, (i = 1, 2, 3, 4, 5)$ lying between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ $such\ that$ $\tan A_i = a_i$.

Divide the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ into four equal intervals, each of length $\frac{\pi}{4}$.

Now, by Pigeon hole Principle at least two of the $A_i s$ must lie in one of the four intervals. Suppose A_k and $A_l with$ $A_k > A_l$ lie in the same interval, then

$$0 < A_k - A_l < \frac{\pi}{4}$$

So tan 0 < tan $(A_k - A_l) <$ tan $\frac{\pi}{2}$

[It is because tan function increases in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$]

$$i.e., 0 < \frac{\tan A_k - \tan A_l}{1 + \tan A_k \tan A_l} < 1$$

$$0 < \frac{a_k - a_l}{1 + a_k a_l} < 1$$

Hence there are two real numbers $x = a_k, y = a_l$ $such\ that$ $0 < \frac{x-y}{1+xy}$.

150. Using $1 = 3^0$, you can weigh $\frac{3^1-1}{2} = 1$ weight is clear. If 1, 3 are given weights, then using both sides of the balance, we can weigh 4 weights as follows.

Wt. of the object	Left pan	Right pan
1	1	Object
3−1= 2	3	Object + 1 (if object's weight is $3 - 1 = 2$.
3	3	Object
3 + 1= 4	3, 1	Object

Thus, we have the four weights 1, 2, 3, 4 $= \frac{3^2-1}{2}$ weighed using only 2 weights, namely, 1 and 3 but using the weights on both pans.

To prove the general case we'll use the principle of Mathematical Induction.

Let us assume that using the weights 1, 3, 3^2, …., 3^k, we can weigh from 1 to $\frac{3^{k+1}-1}{2}$ units.

Now, if we have one more weight, say, 3^{k+1} at our disposal, then we need to prove that we can weigh from 1 unit to $\frac{3^{k+1}-1}{2}$ units. We can clearly weigh from 1 unit to $\frac{3^{k+1}-1}{2}$ units by the first k + 1 weights, namely, 1, 3, 3^2, …, 3^k. We need to prove that additional we can weigh from $\frac{3^{k+1}-1}{2}+1\ to\ \frac{3^{k+2}-1}{2}$ units using the additional weight 3^{k+1} along with others. Consider two cases:

(i) Weighing from $\frac{3^{k+1}-1}{2}+1\ to\ 3^{k+1}-1$.

(i.e.) we want to weigh $\frac{3^{k+1}-1}{2}+r\ where\ 1\leq r\leq\frac{3^{k+1}-1}{2}$. Now for weighing this use the weigh 3^{k+1} in one pan and the object and the weights representing

$$3^{k+1}-\frac{3^{k+1}-1}{2}-r=\frac{3^{k+1}-1}{2}-(r-1)$$

Note that $\frac{3^{k+1}-1}{2}-(r-1)$ can be represented using the weights 1, 3, 3^2, …., 3^k by induction hypothesis:

∴ Using 1, 3, 3^2, …, $3^k, 3^{k+1}$ we can weigh all weights up to $\frac{3^{k+1}-1}{2}+\frac{3^{k+1}-1}{2}=3^{k+1}-1$.

Now to weigh 3^{k+1}, we need to use only one weight, namely, 3^{k+1}.

To weigh from $3^{k+1}+1\ to\ \frac{3^{k+2}-1}{2}, i.e., to\ weigh\ 3^{k+1}+r\ where\ 1\leq r\leq\frac{3^{k+1}-1}{2}\ we\ need\ use\ 3^{k+1}$ weight on one pan along with the representation for the weight r in terms of 1, 3, 3^2, …., 3^k in the same pan and the object in the other pan.

Thus, we can weigh up to $\frac{3^{k+1}-1}{2}$ units starting from 1 unit using the weights 1, 3, 3^2, …, $3^k, 3^{k+1}$.

Note: In all the above weighings we use weights on both the pans, the pan where the object is placed as well as the other pan. This representation is nothing but the balanced ternary representation of rational numbers where the three digits in the balanced ternary representation are taken to be 0, 1, −1 instead of 0, 1, 2. Any number $n\in N$ can be represented as $n=b_0+b_1 3+b_2 3^2+\cdots+b_k 3^k\ where\ b_0, b_1, \dots, b_k\in\{0,1,-1\}$.

151. Let us find the number of times they have to row front and back when there are 1, 2 and 3 couples.

For just one couple, they just row and cross over to the far side thus, it is enough if they row once.

So for $n=1$, no. of times they should row is 1. Observe the following diagrams. For $n=2$;

Let us denote the couples as $(x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)$.

For $n=2$, we have two couples $(x_1,y_1),(x_2,y_2)$.

So, when there are two couples they will reach the far side of the river by rowing 5 times.

When $n=3$, where the couples are (x_1 y_1), (x_2 y_2) and (x_3 y_3) we can infer that when there are n couples they have n_0 row (4n – 3) times. For $n=1$, we say that they row first once which is $4\times1-1=1\ and\ for\ n=2$, we also saw that they rowed 5 times which is $4\times2-3=8-5=5$ is true.

Also again, for $n = 3, we\ get\ 4 \times 3 - 3 = 12 - 3 = 9$ times, to be true.

So, let us see if we can use mathematical induction to see if the formula is true. Since we have already seen that he formula is true for $n = 1$ (also 2, 3). We shall assume that it is true for $n = k$. That is if there are $n = k$ couples, they can cross over the river by rowing (4k – 3)times. Now, draw the diagram from this stage, when there are k + 1 couples.

Thus assuming that they have to row (4k – 3) times when are k couples, we find that they have to row four more times, i.e., (4k + 1) times when there are (k + 1) couples.

$$4k + 1 = 4(k + 1) - 3$$

So, the truth of P(k) implies the truth of P(k + 1) and we have shown that this formula is true for $n = 1$.

Thus, the formula we informed is true for all $n \in D$. So, when there are n couples they have to row (4n – 3) times to row across the river to reach the far side.

152. We will find the total number of contestants.

Since for each pair of problems there were exactly two contestants, let us assume that an arbitrary problem p_1 was solved by r contestants. Each of these r contestants solved 6 more problems, solving 6r more problems in all counting multiplicities. Since every problem, other than p_1, was paired with p_1 and was solved by exactly two contestants, each of the remaining 27 problems (i.e., other than p_1) is counted twice among the problems solved by the r contestants,

$$i.e., 6r = 2 \times 27$$

$$or, r = 9$$

Therefore, an arbitrary problem p_1 is solved by 9 contestants.

Hence, in all we have $\frac{9\times28}{7} = 36$ contestants, as each contestant solves 7 problems.

For the rest of the proof, let us assume the contrary, that is, every contestant solved either 1, 2 or 3 problems in Part I.

Let us assume that there are n problems in part I and let x, y, z be the number of contestants solves either 1, 2 or 3 problems in Part I, we get

$$x + y + z = 36 \dots\dots (1)$$

$$x + 2y + 3z = 9n \dots\dots. (2)$$

(Since each problem was solved by 9 contestants.)

Since every contestant among y solves a pair of problems in Part I and every contestant among z solves 3 pairs of problems in Part I and as each pair of problems was solved by exactly two contestants, we get following equations

$$y + 3z = 2\ n_{C_2} = 2\frac{n(n-1)}{2}$$
$$= n(n-1) \dots. (3)$$

From eq. (1), eq. (2) and eq. (3) we get

$$z = n^2 - 10n + 36$$

$$and\ y = -2n^2 + 29n - 108$$
$$= -2\left(n - \frac{29}{4}\right)^2 - \frac{23}{8} < 0$$

As y < 0 is not an acceptable result, our assumption is wrong. Here, there is at least one contestant who solved either no problem from Part I or solved at least 4 problems from Part I.

153. Since it is given that $1 \in A, 2 \notin A. For\ if\ 2 \in A$, then $2^0 + 2 = 3$ is generated by 2 members of A violating the condition for the partitioning :

$$\therefore 2 \in B$$

Similarly, $3 \notin A\ as\ 1 + 3 = 4 = 2^1 + 2 \quad \therefore 3 \in B$.

But $4 \notin B. For\ if\ 4 \in B, then\ 2^2 + 2 = 4 + 2 = 6$ is generated by two members of B.

$\therefore$ The partitioning for the first few positive integers is

$$A = \{1, 4, 7, 8, 12, 13, 15, 16, 20, 23, \dots\}$$

$$B = \{2, 3, 5, 6, 9, 10, 11, 14, 17, 18, 19, 21, 22, \dots\}$$

Suppose 1, 2, …, n−1 (for $n \geq 3$) have already been assigned to $A \cap B$ in such a way that no distinct members of A or B have a sum $= 2^1 + 2 (l = 0, 1, 2, \dots)$.

Now, we need to assign n to A or B.

Let k be a positive integer such that $2^{k-1} + 2 \leq n < 2^k + 2$.

Then assign 'n' to the complement of the set to which $2^k + 2 - n$ belongs. But for this, we need to check that $2^k + 2 - n$ has already been assigned.

Now as $n \geq 2^{k-1} + 2 > 2^{k-1} + 1$

$$2n > 2^k + 2$$

$$\therefore n > 2^k + 2 - n.$$

Since all numbers below n have been assumed to be assigned to either A or B, $2^k + 2 - n$ has already been assigned and hence n is also assigned uniquely.

For example, consider $k = 1$

$3 = 2^0 + 2 \leq n < 2^1 + 2 = 4$

Consider $n = 3, \quad 4 - n = 1.$

Now $1 \in A$ (given)

$\therefore 3 \in B.$

Consider $k = 2$

$\therefore\ 2^{2-1} + 2 \leq n < 2^2 + 2 = 6$

$4 \leq n < 6$

When $n = 4, as\ 6 - n = 2 \in B$, we assign 4 to A.

When $n = 5, as\ 6 - 5 = 1 \in A$, we assign 5 to B.

Since the set to which n gets assigned is uniquely determined by the set to which $2^k + 2 - n$ belongs, the partitioning is unique.

Looking at the pattern of the partitioning of the initial set of positive integers, we conjecture the following.

(1) $n \in A\ if\ 4|n$
(2) $n \in B\ if\ 2|n\ but\ 4 \nmid n$
(3) If $n = 2^r.k + 1$

($r \geq 1$, k odd, then $n \in A$ if k is of the form 4m – 1 and $n \in B$ if k is of the form (4m + 1).

Proof of the conjecture : We note that 1, 4 ∈ A and 2, 3 ∈ B. If $2^{k-1} + 2 \leq n < 2^k + 2$ and all the numbers less than n have been assigned to A or B and satisfy the above conjecture, then if 4 | n, as $2^k + 2 - n$ is divisible by 2 but not by 4, $2^k + 2 - n \in B$. Hence, $n \in A$. Similarly, if 2 divides n but not 4, then $2^k + 2 - n$ is divisible by 4 and hence, is in A.

$$\therefore n \in B$$

If $n = 2^r.k + 1$

Where r > 1, k odd and $k = 4m - 1$, then

$$2^k + 2 - n = 2^k - 2^r.k + 1$$
$$= 2^r\left(2^{k-r} - k\right) + 1$$

Where clearly $2^{k-r} - k$ is odd and equals 1 (mod 4).

$$\therefore 2^k + 2 - n = 2^k - 2^r.k + 1$$
$$= 2^r\left(2^{k-r} - k\right) + 1$$

Where clearly $2^{k-r} - k$ is odd and equals 1 (mod 4).

$\therefore 2^k + 2 - n \in B.$

Hence, $n \in A$. Similarly, it can be shown that if $n = 2^r.k + 1, where\ k \equiv 1\ (mod\ 4), then\ n \in B$. Thus, the conjecture is proved.

Now, 1988 is divisible by 4.

$\therefore 1988\ \in A$

$$1987 = 2^1.993 + 1\ where\ 993 = 1\ (mod\ 4)$$
$$\therefore 1987 \in B$$

$$1989 = 2^2.497 + 1\ where\ 497 = 1\ (mod\ 4)$$
$$\therefore 1989\ \in B$$

$2|1998 \quad but\ 4'/1998 \quad \therefore 1998 \in B$

$$1997 = 2^2.499 + 1\ where\ 499 = 3\ (mod\ 4)$$
$$\therefore 1997 \in A$$

154. Since each A_i contains 4 elements, totally we get 24 elements of which some may be repeated. But each element is repeated 4 times as each element belongs to exactly 4 of the A_is. Hence there are $\frac{24}{4} = 6$ distinct elements in S.

Since S= $B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_n$, and each B_i consists of 2 elements, this union accounts for 2n elements. But each elements appears exactly 3 times. Thus the number of distinct elements in S is also equal to 2n/3. Therefore $\frac{2n}{3} = 6\ which\ gives\ n = 9$.

155. This makes repeated use of the Pigeon hole Principle. As there are 65 balls and two boxes, one of the boxes must contain at least 33 balls (as otherwise the total number of balls would be $\leq 32 + 32 = 64$).

Consider that box (i.e., the one containing ≥ 33 balls).We have more than 33 balls and four colours (white, black, red, yellow) and hence there must be at least 9 balls of the same colour in the box.

There are at most 4 different sizes available for these 9 balls of the same colour. For if there were 5 (or more) different sizes, then the collection of five balls, all of the different sizes would not satisfy the given property. Thus among these 9 balls (of the same colour and in the same box) there must be at least 3 balls of the same size.

156. Pair off the elements of the set {1, 11, 21, 31, …, 541, 551} as follows : {(1, 551), (11, 541), …{271, 281}}. There are 28 such pairs and they account for all the numbers in the original set.

So if the subset of A has more than 28 elements, then A should contain both the elements of some pair, but then there is a contradiction since each pair above has the property that the two elements in the pair add up to 552. Thus A cannot have more 28 elements.

157. We shall look at the problem form a general viewpoint. For any positive integer, n, let T_n denote the number of permutations of (P_1, P_2, …, P_n) of 1, 2, 3, …, n such that for each k, $1 \leq k \leq n, (P_1, P_2, P_3, \dots, P_n)$ is not a permutation of 1, 2, …, k.

We shall obtain a formula for T_n which expresses T_n in terms of T_1, T_2, …, T_{n-1} (n > 1). (Such a relation is called a recurrence relation for T_n).

Consider any permutation $(P_1, P_2, P_3, \dots, P_n)$ of 1, 2, …, n. there is always a least positive integer k such that $(P_1, P_2, P_3, \dots, P_k)$ is a permutation of 1, 2, .. k. in fact k may be any integer in the set {1, 2, …, n}; and those permutations for which $k = n$ are exactly the ones we wish to count. The number of permutations of (1, 2, …, n) for all of which k is the least positive integer satisfying the above property is T_k . (n – k)!, by our definition of T_n. The second factor corresponds to the permutations of k + 1, k + 2, …, n which fill up the remaining (n – k)places. Since there are n! permutations in all, we obtain

$$n! = \sum_{k-1}^{n} T_k \,.\, (n-k)!$$

$$= T_1 \,.\, (n-1)! + T_2 \,.\, (n-2)! + \cdots + T_{n-1}.\, 1! + T_n.\, 0!$$

Hence

$$T_n = n! - T_1.(n-1)! - T_2.(n-2)! \dots - T_{n-1}.\, 1!$$

Clearly

$$T_1 = 1$$

$$T_2 = 2! - T_1.\, 1! = 2 - 1 = 1$$

$$T_3 = 3! - T_1.\, 1! - T_2.\, 1! = 6 - 2 - 1 = 3$$

$$T_4 = 4! - T_1.\, 1! - T_2.\, 2! = T_3.\, 1!$$

$$= 24 - 6 - 2 - 3 = 13.$$

$$T_5 = 5! - T_1.\, 4! - T_2.\, 3! - T_3.\, 2! - T_4.\, 1!$$

$$= 120 - 24 - 6 - 6 - 13 = 71$$

$$T_6 = 6! - T_1.\, 5! - T_2.\, 4! - T_3.\, 3! - T_4.\, 2! - T_5.\, 1! = 720 - 120 - 24 - 18 - 26 - 71 = 461.$$

Thus the required number is 461.

158. a. Since A contains n + 1 elements of the set {1, 2, 3, …., 2n} some two of the n + 1 element must be consecutive (Why?). But then any two consecutive integers are relatively prime and we have the desired conclusion.

b. We give a proof by making use of the Pigeon hole Principle. Write each of the n + 1 numbers in the form $2^p.\, q$, where q is an odd number and p is a non negative integer. What are the possible values of q? Since the numbers of A come from the set {1, 2, 3, …, 2n}, we see that q can be any one of the n odd numbers 1, 3, 5, 7, …. 2n−1. As there are n + 1 numbers in A, there are n + 1 values of q. Hence by the afore−mentioned principle, for some two numbers $a = 2^{p_1}.\, q_1 \; and \; b = 2^{p_2}.\, q_2$, we must have $q_1 = q_2$.

Since $a \neq b, p_1$ is either greater than p_2 or less than p_2. In the former case b divides a., while in the latter case a divides b.

159. First, not that A has 233 elements of which 116 are even and 117 are odd. B has 42 elements of which 21 are even and 21 are odd and $A \cap B$ has 14 elements.

Therefore, the required number is:

$$n = |\{(a,b) : a \in A, b \in B, a + b \text{ is even}\}| - |\{(a,b) : a \in A, b \in B, a = b\}|$$

$$= |\{(a,b): a \in A, b \in B, a \text{ is even}, b \text{ is even}\}| + |\{(a,b) : a \in A, b \in B, a \text{ odd}, b \text{ odd}\}| - |\{(a,b) : a \in A, b \in B, a = b\}|$$

$$= 116 \times 21 + 117 \times 21 - 14 = 4879.$$

160. Suppose A has r elements, $0 \leq r \leq n$. Such an A can be chosen in $\binom{n}{r}$ ways. For each such A, the set B must necessarily have the remaining (n – r) elements and possibly some elements of A.

Thus, $B = (X \setminus A) \cup C, where\ C \subset A$. Hence B can be chosen in 2^r ways. Thus there are $\sum_{r=0}^{n} \binom{n}{r} 2^r = (1+2)^n = 3^n$ ways of choosing two sets A and B satisfying the given conditions. Among these choices, only in one case $A = B(= X)$, and in all other cases $A \neq B$. Since the order does not matter, we essentially have $(3^n - 1)/2$ pairs.

161. Since x + 1 divides $ax^2 + bc + c$, we must have $a + c = b$.

Thus we have to count the number of triples (a, b, c) with the condition that a, b, c lie in the set {1, 2, 3, …, 1999}, $a \neq c\ and\ a + c = b$. If we take a < c, then for each a with $1 \leq a \leq 999$, c can take values from a + 1 to 1999 – a. Thus for $a = 1$, c runs from 2 to 1998 giving 1997 ordered pairs (a, c) with a < c; for $a = 2$, c runs from 3 to 1997, giving 1995 pairs (a, c) with a < c, and so on.

The number of ordered pairs (a, c) with a < c and a + c lying in the set {1, 2, 3, …, 1999} is thus equal to 1997 + 1995 + 1993+ …+ 1 $= 999^2$.

Similarly the number of pairs (a, c) with c < a and a + c lying in the set {1, 2, 3, …, 1999} is 999^2. Hence the required number of polynomials is $2.999^2 = 1996002$.

162. Let us denote by 0 or 1 the absence or presence of an element of X in the sets A, B, C. For any fixed element of X, there are only four choices to conform with $A \subset B \subset C$, namely, 000, 001, 011, 111. Thus there are 4^nchoices. But $B = C$ gives three choices, namely, 000, 011, 111. Hence there are 3^n triples (A, B, B). The number of triples (A, B, C) with $A \subset B \subset C\ but\ B \neq C$ is therefore $4^n - 3^n$.

163. We fill up the 3×3 array at the left top (shown by dots in the adjacent figure) arbitrarily using the numbers 0, 1, 2, 3. This can be done in 4^9 ways. The three numbers in the first row uniquely fix a. similarly b, c, p, q, r are fixed uniquely (If a number n when divided by 4 leaves a remainder R, then n + (4 – R) is divisible by 4 and 4−R is in the set {0, 1, 2, 3}).

It is also clear that a+b+c and p+q+r leave the same remainder modulo 4, since both are obtained by the same set of nine numbers adding row−wise and adding column−wise, modulo 4. Hence x is also fixed uniquely by the nine numbers originally chosen. Then the number of arrays required is 4^9.

164. Delete any n rows containing maximal number of zeroes. We claim that at most n zeroes are left in the remaining n rows. For, if otherwise, there are at least n + 1 zeroes left and so there are at least 2 zeroes in some row, by the Pigeon hole Principle.

Since we have deleted rows containing maximum number of zeroes, each such row must contain at least 2 zeroes. Hence we would have deleted at least 2n zeroes. These along with n+1 zeroes would account for more than 3n zeroes, a contradiction to the hypothesis. This proves our claim.

Now remove the columns (numbering not more than n) containing the remaining zeroes. By this process, we are removing all the 3n zeroes in the desired manner.

165.

(a) Let a, b, c be the sides of a triangle with a+b+c = 1996, and each being a positive integer.

Then a + 1, b + 1, c + 1 are also sides of a triangle with perimeter 1999 because

$$a < b + c \Rightarrow a + 1 < (b + 1) + (c + 1)$$

and so on. Moreover (999, 999, 1) form the sides of a triangle with perimeter 1999, which is not obtainable in the form (a + 1, b + 1, c + 1) where a, b, c are the integers and the sides of a triangle with a + b +c $= 1996$.

We conclude that f(1999) > f(1996)

(b) As in the case (a) we conclude that f(2000) ≥ f(1997).

On the other hand, if x, y, z are the integer sides of a triangle with $x + y + z = 2000$, and say $x \geq y \geq z \geq 1$, then we cannot have $z = 1$; for otherwise we would get $x + y = 1999$ forcing x, y to have opposite parity so that $x - y \geq 1 = z$ violating triangle inequality for x, y, z. Hence $x \geq y \geq z > 1$. This implies that $x - 1 \geq y - 1 \geq z - 1 > 0$.

If $x \geq y + z - 1$, then we see that y+z−1 ≤ $x < y + z$, showing that $y + z - 1 = 1$. Hence we obtain $2000 = x + y + z = 2x + 1$ which is impossible. We conclude that x < y + z – 1. This shows that (x−1)<(y−1) + (z−1) and hence x−1, y−1, z−1 are the sides of a triangle with perimeter 1997. This gives $f(2000) \leq f(1997)$. Thus we obtain the desired result.