Problem 1

Determine if the following sets are open or closed:

$$TO = \{(x, y) \in \mathbb{R}^2 | 0 < |x - 1| < 1\} \qquad B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq y\}$$
$$VS = \{(x, y) \in \mathbb{R}^2 || x| < 1, |y| \leq 1\} \qquad D = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Q} \text{ and } y \in \mathbb{Q}\}$$
$$E = \{(x, y) \in \mathbb{R}^2 \mid x \notin \mathbb{Q} \text{ Where } y \notin \mathbb{Q}\} \qquad F = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$$

Solution: TO and F are open. B is closed, the others are neither open nor closed. Here is a proof varying the techniques:

1. TO is open. If $(x, y) \in TO$, so $0 < |x - 1| < 1$, which means $x \neq 1$ and $0 < x < 2$. We know then that there is $\varepsilon > 0$ such as $1 \notin]x - \varepsilon, x + \varepsilon[$ and $0 < x - \varepsilon < x + \varepsilon < 2$. Then, $B((x, y), \varepsilon)$ (for the infinite norm) is contained in TO. TO is not closed, because the rest (u_n) defined by $u_n = (1/n, 0)$ is a sequence of elements of TO which converges to $(0, 0)$ who is not in TO.

2. B is closed. Let (x_n, y_n) is a sequence of elements of B which converges to (x, y), then we know that for each integer not, we have $0 \leq x_n \leq y_n$. Going to the limit, we deduce that $0 \leq x \leq y$ and therefore that $(x, y) \in B$. B is not open; in any ball containing $(0, 0)$, there are points which are not in B (the points of the type $(-\varepsilon, 0)$ for example, with $\varepsilon > 0$).

3. VS is not closed, because if $u_n = \left(1 - \frac{1}{n}1\right), (u_n)$ is a sequence of elements of VS which converges to $(1, 1)$ who is not in VS. Also, VS is not open, because any ball containing the point $(0, 1)$, who is in VS, contains elements that are not in VS (for example take the points $(0, 1 + \varepsilon)$.

4. D is not closed: if (r_n) is a sequence of rationales converging to $\sqrt{2}$, then the rest $(r_n, 0)$ is a sequence of elements of D which converges to $(\sqrt{2}, 0)$ which is not part of D. Dis not open. In any center ball $(0, 0)$, which is part of D, there are elements that are not in D, for example elements of the type $\left(0, \frac{\sqrt{2}}{n}\right)$.

5. E is not open because its complementary, D, is not closed. E is not closed because its complementary is not open.

6. F is open because it is the reciprocal image of the open interval $[-\infty, 4]$ by the continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 + y^2$. F is not closed, because the rest (u_n) defined by $u_n = \left(2 - \frac{1}{n}, 0\right)$ is a sequence of elements of D which converges to $(2, 0)$ which is not part of F.

Problem 2

Let $R > 0$. For all integers $n \geq 1$ consider the set

$$S_n(R) = \left\{(x, y) \in \mathbb{R}^2 : \left(x - \frac{1}{n}\right)^2 + \left(y - \frac{1}{n}\right)^2 \leq \frac{R^2}{n^2}\right\}$$

1. Determine whether $S_n(R)$ is open, closed, compact, connected.
2. Determine a condition on R such that $S_{n+1}(R) \subset S_n(R)$.
3. Let $S(R) = \bigcup_{n\geq 1} S_n(R)$. Determine a condition on R such that $S(R)$ is closed.

Solution: 2. It is well known that the center disc A with radius r is contained in the center disc B with radius R if and only if $AB + r \leq R$ (which we can easily convince by drawing a picture). Here, we deduce that $B_{n+1} \subset B_n$ if and only if

$$\sqrt{2\left(\frac{1}{n} - \frac{1}{n+1}\right)^2} \leq \frac{R}{n} - \frac{R}{n+1}.$$

That is to say if and only if $R \geq \sqrt{2}$.

3. If $R \geq \sqrt{2}$, the sets are nested inside each other and $B = B_1$ which is closed. If $R < \sqrt{2}$, then we easily check that $(0, 0)$ is not part of any set B_n. However, the continuation of the centers of B_n, $\left(\frac{1}{n}, \frac{1}{n}\right)$, is contained in B and converges to $(0, 0)$ who is not in B. So B is not closed. Thus, it has been proved that B is closed if and only if $R \geq \sqrt{2}$.

Problem 3

Let E a normalized vector space and F a vector subspace of E. Prove that, if F is open, then $F = E$.

Solution: If F is open, then since $0 \in F$, there exists $r > 0$ such as $B(0, r) \subset F$. But then, let's take $x \in E$, $x \neq 0$. Then $y = \frac{rx}{2\|x\|}$ has for standard $r/2$, so it is an element of F. Since F is stable by multiplication by a scalar, $x = \frac{2\|x\|}{r} y$ is part of F and so $F = E$.

Problem 4

Determine whether the following subsets of $\mathbb{R}$ are open, closed, compact, connected.

$$\mathbb{N}, \quad \mathbb{Z}, \quad \mathbb{Q}, \quad [0, 1), \quad [0, +\infty), \quad (0, 1) \cup \{2\}, \quad \left\{\frac{1}{n} : n \in \mathbb{N}\backslash\{0\}\right\}, \bigcap_{n\geq 1} \left(-\frac{1}{n}, \frac{1}{n}\right)$$

Solution: $\mathbb{N}$ and $\mathbb{Z}$ are closed, because any sequence of natural integers converges to its limit which is a natural integer. Indeed, $0 \in \mathbb{N}$ and no open ball centered in 0 is not contained in $\mathbb{N}$ (an open ball is here an open interval of the type $]-\varepsilon, \varepsilon[$).

Q is neither closed nor open. It is not closed, because for example there exists a sequence of rationales which converges towards the irrational $\sqrt{2}$. It is not open, because for example any open interval centered in 0 contains irrationals. $[0,1[$ is not closed. The following (u_{not}) defined by $u_n = 1 - \frac{1}{n}$ for $n \geq 2$ is contained in $[0,1]$ and converges to 1, but $1 \notin [0,1[.[0,1[$ is not open, because no open interval centered in 0 is not contained in $[0,1[$. For the same reason, $[0,+\infty[$ is not open. On the other hand, $[0,+\infty[$ is closed: for any follow-up (x_n) of positive reals which converges to ℓ, so ℓ is a real positive. $]0,1[\cup\{2\}$ is not closed and is not open either (consider the open intervals centered in 2).

$\left\{\frac{1}{n} : n \in \mathbb{N}\backslash\{0\}\right\}$ is not open (consider open intervals centered in 1), and is not closed either. The following (x_n) defined by $x_n = \frac{1}{n}$ for $n \geq 1$ is contained in this set, but its limit, 0, is not.

Finally, we can notice that $\bigcap_{n\geq 1}\left(-\frac{1}{n}, \frac{1}{n}\right)\{0\}$.. Indeed, if $x \in \bigcap_{n\geq 1}]\left(-1/n, 1/n\right)$, so for everything $n \geq 1$, we have $-\frac{1}{n} \leq x \leq \frac{1}{n}$ and therefore by choosing the limit, $x = 0$. So, this set is closed, but not open (although it is an intersection of open!).

Problem 5

Let $\tilde{d}(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$ for all $A, B \subset X$. State whether the following statements are true or false. Justify your answer. (i) If $A \cap B \neq \emptyset$, then $\tilde{d}(A,B) = 0$. (ii) If $\tilde{d}(A,B) = 0$, then $A \cap B \neq \emptyset$. (iii) $(P(X), \tilde{d})$ is a metric space.

Solution: (i) If $A \cap B \neq \emptyset$, then we can find $a \in A \cap B$ such that $d(a,a) = 0$. This implies that $\tilde{d}(A,B) = 0$. Therefore (i) is true.
(ii) Let $A = (0,1)$ and $B = (1,2)$, $\tilde{d}(A,B) = 0$, but $A \cap B = \emptyset$. Therefore (ii) is false.
(iii) Let $A, B \in P(X)$, assume that $A \neq B$, but $A \cap B \neq \emptyset$. From (i), we will get $\tilde{d}(A,B) = 0$. Therefore $(P(X), \bar{d})$ is not a metric space. (iii) is false.

Problem 6

Let U be an open subset of X, and let $x \in U$. Is $U\backslash\{x\}$ open in X ? Justify your answer.

Solution: We claim that $U\backslash\{x\}$ is open in X. Let $y \in U\backslash\{x\}$ be a arbitrary element. It is easy to observe that $y \in U$. Since U is open subset X, then there exits $r > 0$ such that

$B(y,r) \subset U$. We will have two cases, either $x \notin B(y,r)$ or $x \in B(y,r)$. If $x \notin B(y,r)$ then $B(y,r) \subset U \backslash \{x\}$. If $x \in B(y,r)$ then choose $r_0 = \frac{d(y,x)}{2}$, therefore $B(y, r_0) \subset U \backslash \{x\}$. Hence $U \backslash \{x\}$ is an open set.

Problem 7

Let $f, g : X \to \mathbb{R}_u$ be continuous functions, and let $C = \{x \in X : f(x) = g(x)\}$. Prove that C is closed.

Solution: Consider a sequence of points $\{x_n\} \in C$ converging to $x \in X$. To prove C is closed, it is enough to prove that $x \in C$. Since $f(x_n) = g(x_n)$ for all n and f, g are continuous, then we have

$$f(x) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = g(x).$$

Therefore $x \in C$. Thus C contains its limit points. Hence C is closed.

Problem 8

Let (X, d_1) be a metric space. Define d_2 on $X \times X$ by

$$d_2(x,y) = \begin{cases} d_1(x,y) & \text{if } 0 \leq d_1(x,y) \leq 1 \\ 1 & \text{Otherwise} \end{cases}$$

Show that d_2 is a metric on X. Show that a set A is open in (X, d_1) iff it is open in (X, d_2).

Solution: (i) Let $x, y \in X$, if $d_2(x,y) = 0$ then $d_1(x,y) = 0$, therefore $x = y$.
(ii) Let $x, y \in X$, if $d_1(x,y) \leq 1$ then $d_2(x,y) = d_1(x,y) = d_1(y,x) = d_2(y,x)$. If $d_1(x,y) > 1$ then $d_1(y,x) > 1$, therefore $d_2(x,y) = d_2(y,x)$.
(iii) Let $x, y, z \in X$, If $d_1(x,y) \leq 1$ and $d_1(y,z) \leq 1$ then $d_2(x,y) = d_1(x,y)$ and $d_2(y,z) = d_1(y,z)$. Therefore $d_2(x,z) \leq d_1(x,z) \leq d_1(x,y) + d_1(y,z) = d_2(x,y) + d_2(y,z)$.
If $d_1(x,y) > 1$, then $d_2(x,z) \leq 1 \leq 1 + d_1(y,z) = d_2(x,y) + d_2(y,z)$. Similarly we can prove the other cases. Hence d_2 is a metric on X.

Let U be d_1-open and $x \in U$, then there exits $r > 0$ such that $B_{d_1}(x,r) \subset U$. If this holds for r, then it holds for any $0 < \epsilon < r$. So we may assume that $0 < r < 1$. In such case $B_{d_1}(x,r) = B_{d_2}(x,r)$, so that $B_{d_2}(x,r) \subset U$. Therefore U is d_2 - open. Similarly we can prove that if U is d_2-open then U is d_1-open.

Problem 9

Define $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ by $d(m,n) = \left|\frac{1}{m^2} - \frac{1}{n^2}\right|$. Show that d defines a metric on $\mathbb{N}$. Show that $\mathbb{N}$ complete with respect to this metric.

Solution: (i) If $d(m,n) = 0$ then $\left|\frac{1}{m^2} - \frac{1}{n^2}\right| = 0, m^2 = n^2$, implies $m = n$.
(ii) $d(m,n) = \left|\frac{1}{m_1^2} - \frac{1}{n^2}\right| = \left|\frac{1}{n_1^2} - \frac{1}{m^2}\right| = d(n,m)$.
(iii) $d(m,n) = \left|\frac{1}{m^2} - \frac{1}{n^2}\right| = \left|\frac{1}{m^2} - \frac{1}{k^2} + \frac{1}{k^2} - \frac{1}{n^2}\right| \leq \left|\frac{1}{m^2} - \frac{1}{k^2}\right| + \left|\frac{1}{k^2} - \frac{1}{n^2}\right| = d(m,k) + d(k,n)$.
Therefore d is metric.

Consider a sequence $(x_n = n)$, we will prove that (x_n) is Cauchy sequence. Let $\epsilon > 0$. Since $\frac{1}{n^2}$ converges to 0 with respect to the usual topology in $\mathbb{R}$, we have the following assertion:

$$\exists\, n_0 \in \mathbb{N} \text{ such that } \forall\, m,n \ \geq \ n_0, \quad \frac{1}{n^2} < \epsilon \text{ and } \frac{1}{m^2} < \epsilon.$$

Now $\forall\, m,n \ \geq \ n_0$, we have

$$d(n,m) = \left|\frac{1}{n^2} - \frac{1}{m^2}\right| \leq \left|\frac{1}{n^2}\right| + \left|\frac{1}{m^2}\right| < \epsilon + \epsilon = 2\epsilon$$

Thus (x_n) is Cauchy sequence.
Now, we will prove that (x_n) does not converges to a, for all $a \in \mathbb{N}$. Since $d\left(x_n, a\right) = \left|\frac{1}{n^2} - \frac{1}{a^2}\right| \to \frac{1}{a^2} \neq 0$, therefore x_n does not converges to a.

Problem 10

On $\mathbb{R}^2$, consider the usual metric d, defined by

$$d\left(\left(x_1, x_2\right), \left(y_1, y_2\right)\right) = \sqrt{\left(x_1 - y_1\right)^2 + \left(x_2 - y_2\right)^2}$$

For the following subsets A, B of $\mathbb{R}^2$, determine the closure and the interior of the closure.
(i) $A = \{(x_1, x_2) : x_1 + x_2 \text{ is rational }\}$; (ii) $B = \{(x_1, x_2) : |x_1| < |x_2|\}$.

Solution: (i) Since $\mathbb{Q} \times \mathbb{Q} \subset A$, we have $\mathbb{R}^2 = \overline{\mathbb{Q}} \times \mathbb{Q} \subset \bar{A}$. Therefore $\bar{A} = \mathbb{R}^2$. Since $\bar{A} = \mathbb{R}^2$ is open in $\mathbb{R}^2$, we have $(\bar{A})^o = \left(\mathbb{R}^2\right)^o = \mathbb{R}^2$.

(ii) Let $S = \{(x_1, x_2) : |x_1| \leq |x_2|\}$, we claim that $\bar{B} = S$. Let $x = (x_1, x_2) \in S$, if $|x_1| < |x_2|$ then $x \in B$, therefore $x \in \bar{B}$. Suppose that $|x_1| = |x_2|$ and let $r > 0$ arbitrary. We will show that $B(x,r) \cap B \neq \emptyset$. For this, choose $0 < \delta < r$ such that the closed rectangle $[x_1 - \delta, x_1 + \delta] \times [x_2 - \delta, x_2 + \delta] \subset B(x,r)$. Now, take $y = (x_1 - \delta, x_2)$, since

$|x_1 - \delta| < |x_1| = |x_2|$, we have $y \in B$. Clearly $y \in B(x,r)$ and therefore $B(x,r) \cap B \neq \emptyset$. This implies that $x \in \bar{B}$. Thus $S \subset \bar{B}$. We know that S is closed in $\mathbb{R}^2$ and contains B, therefore $\bar{B} \subset S$. Hence $\bar{B} = \{(x_1, x_2) : |x_1| \leq |x_2|\} = S$.

Now we claim that $(\bar{B})^o = B$. Since B is open, we have $B = B^o \subset (\bar{B})^o$. If $x = (x_1, x_2)$ with $|x_1| = |x_2|$ then for any $r > 0$, we can choose $0 < \delta < r$ such that the closed rectangle $[x_1 - \delta, x_1 + \delta] \times [x_2 - \delta, x_2 + \delta] \subset B(x,r)$. Now, take $y = (x_1 + \delta, x_2)$. Clearly $y \in B(x,r)$, but $y \notin \bar{B}$ Thus, for every $r > 0$, we have $B(x,r) \not\subset \bar{B}$. This implies that $\bar{B}^o = B = \{(x_1, x_2) : |x_1| < |x_2|\}$.