Topology and Analysis 1

Tutorial Worksheet 1 - Metric and normed spaces

Exercise 1. Draw the following sets on the Euclidean plane $\mathbb{R}^2$:

- $B_1^1(\mathbf{0}) = \left\{ \mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_1 < 1 \right\};$
- $B_1^2(\mathbf{0}) = \left\{ \mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 < 1 \right\};$
- $B_1^\infty(\mathbf{0}) = \left\{ \mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_\infty < 1 \right\}.$

Exercise 2. Consider a square matrix $\mathbb{A} \in \mathbb{R}^{n\times n}$.

1. Prove that, if $\det \mathbb{A} \neq 0$, the functions

   $$F_1(\mathbf{x}) = \|\mathbb{A}\mathbf{x}\|_1, \quad F_2(\mathbf{x}) = \|\mathbb{A}\mathbf{x}\|_2 \quad \text{and} \quad F_\infty(\mathbf{x}) = \|\mathbb{A}\mathbf{x}\|_\infty, \quad \mathbf{x} \in \mathbb{R}^n$$

   are norms on $\mathbb{R}^n$.
2. What happens if $\det \mathbb{A} = 0$?

Exercise 3. Denote by $C^0([a,b])$ the vector space of real-valued continuous functions defined on an interval $[a,b]$.

1. Prove that

   $$\|f\|_1 = \int_a^b |f(x)| dx \quad \text{and} \quad \|f\|_\infty = \max_{x\in[a,b]} |f(x)|, \quad f \in C^0([a,b])$$

   are norms on $C^0([a,b])$.
2. Find $C > 0$ such that

   $$\|f\|_1 \leq C\|f\|_\infty, \quad f \in C^0([a,b]).$$

Exercise 4. Denote by $C^1([a,b])$ the vector space of real-valued continuously differentiable functions defined on an interval $[a,b]$.

1. Prove that

   $$\|f\|_a = |f(a)| + \|f'\|_\infty \quad \text{and} \quad \|f\|_b = \|f\|_\infty + \|f'\|_\infty, \quad f \in C^1([a,b])$$

   are norms on $C^1([a,b])$.
2. Show that $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent.
3. Is the function $F : C^1([a,b]) \to [0,+\infty)$ defied as

   $$F(f) = \|f'\|_\infty$$

   a norm on $C^1([a,b])$? Motivate the answer.

Exercise 5. Consider the set of natural numbers $\mathbb{N} = \{0,1,2,\ldots\}$. Consider the function $d : \mathbb{N} \times \mathbb{N} \to [0,+\infty)$ defined as

$$d(n,m) = \begin{cases} \left|\dfrac{1}{n} - \dfrac{1}{m}\right|, & n \neq 0, \quad m \neq, 0 \\ \left|\dfrac{1}{n}\right|, & n = 0, \quad m \neq 0, \\ \left|\dfrac{1}{m}\right|, & n \neq 0, \quad m = 0, \\ 0, & n = 0, \quad m = 0. \end{cases}$$

Prove that d is a distance.

Exercise 6. Let X be a nonempty set. Consider the function $d : X \times X \to [0,+\infty)$ defined as

$$d(x,y) = \begin{cases} 1, & x \neq y, \\ 0, & x = y. \end{cases}$$

Prove that d is a distance (*discreet distance*).