Tutorial Worksheet 2 - Elements of topology

Problem 1. Consider the set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. Consider the function $d : \mathbb{N} \times \mathbb{N} \to [0, +\infty)$ defined as

$$d(n,m) = \begin{cases} \left|\dfrac{1}{n} - \dfrac{1}{m}\right|, & n \neq 0, \quad m \neq, 0 \\ \left|\dfrac{1}{m}\right|, & n = 0, \quad m \neq 0, \\ 0, & n = 0, \quad m = 0. \end{cases}$$

1. Prove that d is a distance.
2. Draw X as a subset of the real line.
3. Find the interior and the accumulation points of X.
4. Say if X is open, closed, compact, connected.

Problem 2. Let X be a nonempty set. Consider the function $d : X \times X \to [0, +\infty)$ defined as

$$d(x,y) = \begin{cases} 1, & x \neq y, \\ 0, & x = y. \end{cases}$$

1. Prove that d is a distance (*discreet distance*).
2. Draw X in the case when it has, respectively, 1, 2, 3 and 4 elements.
3. Find the interior and the accumulation points of X.
4. Say if X is open, closed, compact, connected.

Problem 3. Determine whether the following subsets of $\mathbb{R}$ are open, closed, compact, connected.

$$\mathbb{N}, \quad \mathbb{Z}, \quad \mathbb{Q}, \quad [0,1), \quad [0,+\infty), \quad (0,1) \cup \{2\}, \quad \left\{\frac{1}{n} : n \in \mathbb{N} \setminus \{0\}\right\}, \quad \bigcap_{n \geq 1} \left(-\frac{1}{n}, \frac{1}{n}\right).$$

Problem 4. Determine whether the following subsets of $\mathbb{R}^2$ are open, closed, compact, connected.

$$A = \left\{(x,y) \in \mathbb{R}^2 : 0 < |x-1| < 1\right\}, \quad B = \left\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq y\right\},$$

$$C = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4\right\}, \quad D = \left\{(x,y) \in \mathbb{R}^2 : y \leq x^2\right\},$$

$$E = \left\{(x,y) \in \mathbb{R}^2 : x \in \mathbb{Q}, y \in \mathbb{Q}\right\}, \quad F = \left\{(x,y) \in \mathbb{R}^2 : x \notin \mathbb{Q}, y \notin \mathbb{Q}\right\}.$$

Problem 5. Let $R > 0$. For all integers $n \geq 1$ consider the set

$$S_n(R) = \left\{(x,y) \in \mathbb{R}^2 : \left(x - \frac{1}{n}\right)^2 + \left(y - \frac{1}{n}\right)^2 \leq \frac{R^2}{n^2}\right\}.$$

1. Determine whether $S_n(R)$ is open, closed, compact, connected.
2. Determine a condition on R such that $S_{n+1}(R) \subset S_n(R)$.

3. Let $S(R) = \bigcup_{n\geq 1} S_n(R)$. Determine a condition on R such that $S(R)$ is closed.

Problem 6. Let $(V, \|\cdot\|)$ be a normed vector space on $\mathbb{R}$ or $\mathbb{C}$, and let W be a subspace of V. Prove that, if W is open, then $W = V$.

Problem 7. Let (X, d) be a metric space. Prove the following statements.

1. If $\{A_\gamma\}_{\gamma\in\Gamma}$ is a family of open sets in X, then: $\bigcup_{\gamma\in\Gamma} A_\gamma$ is open.
2. If $\{A_i\}_{i=1,\cdots,n}$ is a *finite* family of open sets in X, then: $\bigcap_{i=1}^{n} A_i$ is open.
3. Show, providing a suitable counterexample, that an infinite intersection of open sets is not necessarily open.
4. What is possible to say about unions and intersections of closed sets?

   [*Hint*: use the De Morgan's duality laws: $\left(\bigcup_{\gamma\in\Gamma} A_\gamma\right)^c = \bigcap_{\gamma\in\Gamma} A_\gamma^c$ and $\left(\bigcap_{\gamma\in\Gamma} A_\gamma\right)^c = \bigcup_{\gamma\in\Gamma} A_\gamma^c$]