Tutorial Worksheet 3 - Series of Functions

Problem 1. Establish if the following series of functions $\sum_{n=1}^{+\infty} f_n$ converge uniformly on the specified interval I.

1. $I = [-1, 1]$ and $f_n(x) = \dfrac{x^n}{n^3}$.
2. $I = \mathbb{R}$ and $f_n(x) = \dfrac{\sin(nx)}{n^2 + 1}$.
3. $I = [0, +\infty)$ and $f_n(x) = x^n e^{-nx}$.
4. $I = \mathbb{R}$ and $f_n(x) = \dfrac{x^n}{n!}$.
5. $I = \mathbb{R}$ and $f_n(x) = \dfrac{1}{n^2 + x^2}$.
6. $I = [-0.9, 0.9]$ and $f_n(x) = nx^n$.

Problem 2. Use suitable upper bounds to prove that the following series of functions $\sum_{n=1}^{+\infty} f_n$ converge totally.

1. $I = [1, +\infty)$ and $f_n(x) = \left(\dfrac{\ln x}{x}\right)^n$.
2. $I = [1/10, +\infty)$ and $f_n(x) = \dfrac{x}{1 + n^2 x^2}$.

Problem 3. For all $n \geq 1$ we consider the sequence of functions $u_n : \mathbb{R} \to \mathbb{R}$ defined as

$$u_n(x) = \ln\left(1 + \frac{x^2}{n^2}\right).$$

1. Determine the domain of point-wise convergence of the series $\sum_{n=1}^{+\infty} u_n(x)$ and $\sum_{n=1}^{+\infty} u'_n(x)$.
2. Show that $\sum_{n=1}^{+\infty} u'_n(x)$ does not converge totally on $\mathbb{R}$ (*Hint*: compute $u'_n(x)$).
3. Show that for all $a > 0$ the series $\sum_{n=1}^{+\infty} u'_n$ converges totally on $(-a, a)$.
4. What is possible to deduce from the previous points?

Problem 4. Show that for all $x > 0$ the series $\sum_{n=1}^{+\infty} e^{-n^2 x}$ converges point-wise, and denote by $S(x)$ its sum. Prove that $S \in C^\infty((0, +\infty))$.

Problem 5. Let $f_n(x) = \dfrac{x^n}{1 + x^n}$. Show that the series $\sum_{n=1}^{+\infty} f_n$ converges totally on $[0, c]$ for all $0 < c < 1$. Show that the convergence is not uniform on $[0, 1)$.

Problem 6. (Uniform d'Alembert criteria) Let $I \subset \mathbb{R}$ be an interval and $f_n : I \to \mathbb{C}$. Assume the following properties:

a) for all $n \in \mathbb{N}$ f_n is bounded on I;

a) for some $n \in \mathbb{N}$ f_n has zeroes in I;

a) there exists $N \in \mathbb{N}$ and $\ell \in [0,1)$ such that $\left|\dfrac{f_{n+1}(x)}{f_n(x)}\right| \leq \ell < 1.$ for all $n \geq N$.

Prove that $\displaystyle\sum_{n=1}^{+\infty} f_n$ converges totally on I.

Problem 7. We study the series of function defined as

$$\sum_{n=1}^{+\infty} \frac{\cos(n\vartheta)}{n}.$$

We use a criteria due to Abel, which consists into studying preliminary $\sum_{n=1}^{+\infty} \dfrac{\cos(n\vartheta)}{n} r^n$.

1. Let $\vartheta \in \mathbb{R}$ to be fixed. Show that the series of functions (in the variable r)

$$\sum_{n=1}^{+\infty} \cos(n\vartheta) r^{n-1}$$

   converges totally on $[-R, R]$ for any $0 < R < 1$.

2. Let $\vartheta \in \mathbb{R}$ to be fixed. Show that the series of functions (in the variable r)

$$\sum_{n=1}^{+\infty} \frac{\cos(n\vartheta)}{n} r^n$$

   converges totally on $[-R, R]$ for any $0 < R < 1$.

3. Let $\vartheta \in \mathbb{R}$ to be fixed. Use the complex number representation to show that for all $|r| < 1$ the following identity holds

$$\frac{\cos\vartheta - r}{1 - 2r\cos\vartheta + r^2} = \cos\vartheta + r\cos(2\vartheta) + r^2\cos(3\vartheta) + \cdots$$

4. Let $\vartheta \in \mathbb{R}$ to be fixed. Deduce that

$$-\frac{1}{2}\ln\left(1 - 2r\cos\vartheta + r^2\right) = \sum_{n=1}^{+\infty} \frac{\cos(n\vartheta)}{n} r^n$$

   for all $|r| < 1$.

5. Fix $|r| < 1$ and consider the series of functions (in the variable ϑ)

$$\sum_{n=1}^{+\infty} \frac{\cos(n\vartheta)}{n} r^n.$$

   Show that it converges totally on $\mathbb{R}$.

6. Use the identity above to compute the value of the integral

$$I_n = \int_0^{\pi} \ln\left(1 - 2r\cos\vartheta + r^2\right) d\vartheta$$

   for all $|r| < 1$.

Problem 8. Consider the sequence of functions $f_n(x) = (n+1)\cos^n(x)\sin(x)$ defined on $\mathbb{R}$.

1. Determine the point-wise limit $f(x) = \lim_{n\to+\infty} f_n(x)$.
2. Show that

$$\lim_{n\to+\infty} \int_0^{\pi/2} f_n(x)dx \neq \int_0^{\pi/2} f(x)dx.$$

3. Show that the series $\sum_{n=0}^{+\infty} f_n$ converges point-wise on $\mathbb{R}$.

4. Show that for all $\varepsilon > 0$, the series $\sum_{n=0}^{+\infty} f_n$ converges totally on the interval $[\varepsilon, \pi/2]$. Deduce that the associated function $f = \sum_{n=0}^{+\infty} f_n$ is continuous on $(0, \pi/2]$.

5. Show that f is not continuous in 0. (*Hint*: find a primitive F_n of f_n and study the function $\sum_{n=0}^{+\infty} F_n$).

6. Does the series $\sum_{n=0}^{+\infty} f_n$ converge uniformly on $[0, \pi/2]$?

Problem 9. For all $s \in \mathbb{C}$, we define the *Riemann Zeta Function*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}.$$

1. Show that $\zeta(s)$ converges if $\operatorname{Re} s > 1$.

2. Show that $\zeta \in C^\infty((1, +\infty))$.

3. Show, using a comparison with an integral, that $\lim\limits_{x \to +\infty} \zeta(x) = 1$.

4. Show that $\lim\limits_{x \to 1^+} \zeta(x) = +\infty$.

Advanced problems

Problem 10. Let $0 < \delta < 1$. Show that the series of functions (in the variable ϑ)

$$\sum_{n=1}^{+\infty} \frac{e^{in\vartheta}}{n}$$

converges uniformly on the interval $(\delta, 2\pi - \delta)$. Is the convergence also total?

Problem 11. (Dirichlet series) Let $\{a_n\}_{n \geq 1}$ be a complex sequence. For all $x \in \mathbb{R}$ we define

$$L(x) = \sum_{n=1}^{+\infty} \frac{a_n}{n^x} \quad \text{(\textit{Dirichlet series} associated to } \{a_n\}_{n\geq 1}).$$

1. Suppose that $L(x_0)$ converges for some $x_0 \in \mathbb{R}$. Show that L converges uniformly on $[x_0, +\infty)$.

2. Suppose that $\{a_n\}_{n \geq 1}$ is periodic of period $p \geq 2$ and $a_1 + a_2 + \cdots + a_p = 0$. Show that for all $x > 0$, $L(x)$ converges and that the function $L : (0, +\infty) \to \mathbb{C}$ is continuous. Deduce that

$$x \mapsto \sum_{n=1}^{+\infty} \frac{\sin(n\pi/8)}{n^x}$$

defines a continuous function on $(0, +\infty)$.