**Tutorial Worksheet 4 - Power Series**

**Problem 1.** Let $S(x) = \sum a_n z^n$ be a complex coefficients power series.

1. Recall the notion of radius of convergence $\rho$.
2. Show that if $|z| < \rho$, then $S(z)$ converges, and if $|z| > \rho$, then $S(z)$ diverges.
3. Let $r > 0$ such that $S(z)$ converges if $|z| < r$, and $S(z)$ diverges if $|z| > r$. Show that $r = \rho$.

**Problem 2.** (Calculating the radius of convergence) Find the radius of convergence of the following power series $\sum a_n z^n$.

1. $a_n = q^n, \quad q \in \mathbb{C} \setminus \{0\}$.
2. $a_n = \dfrac{1}{n^2} \quad (n \geq 1)$.
3. $a_n = \dfrac{1}{n^{100}} \quad (n \geq 1)$.
4. $\lim_{n\to+\infty} a_n = \ell, \quad \ell \neq 0$.
5. $a_n = \dfrac{n^n}{n!}$.
6. $a_n = (\ln n)^{-\ln n} \quad (n \geq 3)$.
7. $a_n = \cos(2\pi n/5)$.
8. $a_n = \dfrac{\sin(\pi n/3)}{5^n(2n+1)}$.

**Problem 3.** Let $\sum a_n z^n$ be a power series with radius of convergence $\rho$.

1. Let $k \in \mathbb{N} \setminus \{0\}$. Show that the radius of convergence of $\sum a_n z^{kn}$ is $\sqrt[k]{\rho}$.
2. Determine the radius of convergence of $\displaystyle\sum_{n=0}^{+\infty} \frac{z^{3n}}{2^n}$ and $\displaystyle\sum_{n=0}^{+\infty} \frac{z^{5n}}{(n+3)3^n}$.

**Problem 4.** (Theoretical questions on the radius of convergence) Say of the following statements are true or false (for each one provide a proof or a counter-example).

1. The series $\sum a_n z^n$ and $\sum a_n(-1)^n z^n$ have the same radius of convergence.
2. The series $\sum a_n z^n$ and $\sum a_n(-1)^n z^n$ have the same domain of convergence.
3. If the the series $\sum a_n x^n$ $(x \in \mathbb{R})$ has infinite radius of convergence, then it converges totally on $\mathbb{R}$.
4. If the the series $\sum a_n x^n$ $(x \in \mathbb{R})$ has finite radius of convergence $R > 0$, and we denote by $f(x)$ its sum, then $\lim\limits_{x\to R^-} f(x)$ and $\lim\limits_{x\to -R^+} f(x)$ exist.

**Problem 5.** This is a variation of Problem 1. Let $\sum a_n z^n$ be a power series with radius of convergence $\rho$.

1. Show that if $|z| < \rho$, then $\{a_n z^n\}$ is bounded. On the other hand, show that if $\{a_n z^n\}$ is bounded, then $|z| \leq \rho$.
2. Let $r > 0$ be such that $\{a_n z^n\}$ is bounded if $|z| < r$ and $\{a_n z^n\}$ is unbounded if $|z| < r$. Show that $r = \rho$.

**Problem 6.** (Sum of power series) Determine the radius of convergence of the following power series, and describe those sums in terms of elementary functions.

1. $A(x) = \sum_{n=0}^{+\infty} n^2 x^n$.
2. $B(x) = \sum_{n=0}^{+\infty} \frac{x^n}{(n+1)!}$.
3. $L(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n}$.
4. $S(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n+2}$.
5. $T(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n+k}$, with $k \in \mathbb{N} \setminus \{0\}$.
6. $T(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n(n+2)}$. [*Hint*: write $\frac{1}{n(n+2)}$ as $\frac{a}{n} + \frac{b}{n+2}$.]

**Problem 7.** (Representation in power series) Show that the function $f : \mathbb{R} to \mathbb{R}$ defines as $f(x) = \frac{x}{1+x^2}$ con be represented as a power series on the interval $(-1, 1)$.

**Problem 8.** (Fibonacci sequence) Consider the sequence

$$1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad \cdots,$$

that is, the sequence $\{F_n\}$ defined by the recursive relations

$$F_0 = F_1 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \quad n \geq 2.$$

A. de Moivre showed a connection of the above sequence and the *golden ratio* $\frac{1+\sqrt{5}}{2}$. The aim of this exercise is the one to exploit this relation using power series.

1. We define $S(x) = \sum_{n=0}^{+\infty} F_n x^n$. Show that its radius of convergence is $\geq \frac{1}{2}$. [*Hint*: show, by induction, that $F_n \leq 2^n$].
2. Show that for all $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, there holds
$$(1 - x - x^2) S(x) = 1.$$
3. Consider the roots of the polynomial $x^2 + x - 1$
$$\alpha = \frac{-1+\sqrt{5}}{2} \quad \text{and} \quad \beta = -\frac{1+\sqrt{5}}{2}.$$
Find $a, b \in \mathbb{R}$ such that
$$\frac{1}{x^2+x-1} = \frac{a}{x-\alpha} + \frac{b}{x-\beta}.$$
4. Show that
$$F_n = \frac{1}{\sqrt{5}} \left(\varphi^{n+1} - \psi^{n+1}\right),$$
where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.
5. What is the radius of convergence of $S$?

**Power series and differential equations**

**Problem 9.** (Weber equation) Let $k \in \mathbb{R}$. Show that the differential equation

$$y''(x) - xy'(x) + ky(x) = 0$$

admits a unique solution $\varphi : \mathbb{R} \to \mathbb{R}$, which can be represented by a power series with radius of convergence $\rho = +\infty$ and that fulfills the conditions $\varphi(0) = 1$ and $\varphi'(0) = 0$.

**Problem 10.** (Legendre equation) Let $\lambda \in \mathbb{C}$. Consider the differential equation

$$(L) \qquad (1-x^2)y''(x) - 2xy'(x) + \lambda y(x) = 0.$$

1. Let $\varphi(x) = \sum_{n=0}^{+\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. Show that $\varphi$ is a solution of $(L)$ if and only if its coefficients $a_n$ fulfill a recursive relation to be determined.
2. Suppose that $\lambda \notin \mathbb{N}$. Let $\varphi(x) = \sum_{n=0}^{+\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ which is a solution of $(L)$. Suppose that $\varphi(0) = 1$ and $\varphi'(0) = 0$. Compute $R$.
3. Suppose $\lambda = \ell(\ell+1)$, with $\ell \in \mathbb{N}$, $\ell \geq 1$.
   (a) Assume that $\ell$ is even. Show that $(L)$ has a polynomial solution $\varphi$ such that $\varphi(0) = 1$ and $\varphi'(0) = 0$.
   (b) Assume that $\ell$ is odd. Show that $(L)$ has a polynomial solution $\varphi$ such that $\varphi(0) = 0$ and $\varphi'(0) = 1$.

**Problem 11.** (Hyper-geometric equation) Let $\alpha, \beta, \gamma \in \mathbb{C}$, and suppose that $\gamma$ is not a negative integer. Consider the differential equation

$$(H) \qquad (z - z^2)y''(z) - [\gamma - (\alpha+\beta+1)z]y'(z) - \alpha\beta y(z) = 0.$$

1. We assume that the solution to $(H)$ can be represented by a power series $S(z) = \sum_{n=0}^{+\infty} a_n z^n$ be a power series with radius of convergence $\rho > 0$. Show that
$$a_{n+1} = \frac{(n+\alpha)(n+\beta)}{(n+1)(n+\gamma)} a_n.$$
2. Show that $(H)$ admits a solution $S(z) = \sum_{n=0}^{+\infty} a_n z^n$ with radius of convergence $\rho > 0$ fulfilling $S(0) = 1$. Compute $\rho$. Such a solution is called the *Hyper-geometric series*, and denoted by $F(\alpha, \beta, \gamma, z)$.
3. The following exercise shows how to obtain any usual function by means of a Hyper-geometric series $F(\alpha, \beta, \gamma, z)$..
   (a) Show that, for all $n \geq 1$, we have
   $$a_n = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\cdot\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)\cdot n!}.$$
   (b) Let $\nu \in \mathbb{R}$ and $|Z| < 1$. Show that $(1-z)^\nu = F(-\nu, \frac{1}{2}, \frac{1}{2}, z)$.
   (c) Let $x \in (0,1)$. Show that $\ln(1-x) = -xF(1,1,2,x)$.
   (d) For all $c \in \mathbb{C}$, show that $e^c = \lim_{\lambda \to 0} F(1, \frac{1}{\lambda}, 1, \lambda c) \quad (|\lambda| < \frac{1}{|c|})$.

**Problem 12.** (Bessel series) Consider the differential equation

$$(B_\alpha) \qquad x^2y''(x) + xy'(x) + (x^2 - \alpha^2)y(x) = 0.$$

1. Show that $(B_0)$ admits a solution $J : \mathbb{R} \to \mathbb{R}$, which can be represented by a power series with radius of convergence $\rho = +\infty$ and that fulfills the conditions $J(0) = 1$ and $J'(0) = 0$.
2. We aim at generalizing the previous result. Let $\alpha \in \mathbb{R}$, and suppose it is not a negative integer. Show that $(B_{0\alpha})$ admits a solution of the form $x^\alpha\phi(x)$, where $\phi$ can be represented by a power series with radius of convergence $\rho = +\infty$ and that fulfills the conditions $\phi(0) = 1$ and $\phi(0) = 0$. [*Hint*: determine the differential equation that is fulfilled by $\phi$.]

**Problem 13.** Consider the Cauchy problem

$$\begin{cases} x^2 y''(x) + (3x - 1) y'(x) + y(x) = 0, \\ y(0) = y'(0) = 1. \end{cases}$$

Is it possible to find a solution $\phi$ which can be represented as a power series with strictly positive radius of convergence?