Computation of Fourier coefficients

Problem 1. Consider the 2π-periodic function f defined on the $[0, 2\pi)$ as

$$f(x) = x, \quad x \in [0, 2\pi).$$

1. Find the complex Fourier coefficients of f.
2. Consider the 2π-periodic function g defined on the $[0, 2\pi)$ as

$$g(x) = \frac{\pi - x}{2}, \quad x \in [0, 2\pi).$$

   Find the complex Fourier coefficients of g, and deduce its trigonometric coefficients.
3. Write the partial sum $S_N(x)$ of g.
4. Show that, for all $x \in (0, 2\pi)$, there holds $\displaystyle\sum_{n=1}^{+\infty} \frac{\sin(nx)}{n} = \frac{\pi - x}{2}$.
5. Show that $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.
6. Consider $G(x) = \int_0^x g(t)dt$. Show that $G \in \mathcal{M}^1 \cap C^0(\mathbb{R})$.
7. Compute the Fourier coefficients of G.
8. Show that $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

Problem 2. Consider the 2π-periodic function f defined on the $[-\pi, \pi)$ as

$$f(x) = x, \quad x \in [-\pi, \pi).$$

1. Find the complex Fourier coefficients of f, and deduce its trigonometric coefficients.
2. Compute the sum of the series $\displaystyle\sum_{n=1}^{+\infty} (-1)^n \frac{\sin(nx)}{n}$.
3. Show Leibniz formula: $\displaystyle\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$.

Problem 3. Consider the 2π-periodic function f defined on the $[-\pi, \pi)$ as

$$f(x) = |x|, \quad x \in [-\pi, \pi).$$

1. Find the complex Fourier coefficients of f, and deduce its trigonometric coefficients.
2. Show that $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$ and $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$.