Ctanujit Classes of Mathematics,
Statistics & Economics.
Kolkata, W. B.,8420253573
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TIFR PREVIOUS YEARS SOLVED PAPERS

Previous Years Papers are solved here. Topic wise Solutions are given so that it gives more comfort while reading this book .

QUESTIONS ARE MOSTLY TAKEN FROM THE FOLLOWING TOPICS :

Number Theory , Theory Of Equations , Linear Algebra , Abstract Algebra , Real Analysis , Topology , Set Theory , Combinatorics & Probability , Differential Equations.

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Q.1. [TIFR'2010]
The last digit of 2^{80} is

(a) 2 (b) 4 (c) 6 (d) 8

Solution:-

Last digits:

$2^1 \equiv 2 \pmod{10}$
$2^2 \equiv 4 \pmod{10}$
$2^3 \equiv 8 \pmod{10}$
$2^4 \equiv 6 \pmod{10}$
$2^5 \equiv 2 \pmod{10}$

$2^{80} = (2^{25\times 3}) \cdot 2^5$
$= (2^{25})^3 \cdot 2^5$
$= (2)^3 \cdot 2$, last digit
$= 6$, last digit.

Q.2. [TIFR'10]
The sum of the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{100\cdot 101} \text{ is}$$

(a) $\frac{99}{101}$ (b) $\frac{98}{101}$ (c) $\frac{99}{100}$ (d) None

Solution:- (d)

$$S = \frac{2-1}{1\cdot 2} + \frac{3-2}{2\cdot 3} + \cdots + \frac{101-100}{100\cdot 101}$$

$$= 1 - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}} - \cdots + \cancel{\tfrac{1}{100}} - \frac{1}{101}$$

$$= \frac{100}{101}.$$

Q.3. [TIFR'2010] The maximum value of $f(x) = x^n(1-x)^n$ for a natural number $n \geq 1$ and $0 \leq x \leq 1$ is

(a) $1/2^n$ (b) $1/3^n$ (c) $1/5^n$ (d) $\frac{1}{4^n}$.

Solution:- (d) For maximum take $x = (1-x) \Rightarrow x = 1/2$.

[If $a+b = \lambda (>0)$ given then ab is max. when $a=b$]

So, $\max\{x^n(1-x)^n\} = \left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^n = \frac{1}{4^n}$.

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Q.4. [TIFR'2011] The equation $x^{22} \equiv 2 \pmod{23}$ has

(a) no solution (b) 23 solutions (c) exactly one soln. (d) 22 soln.

Solution:- (a) $x^{22} \equiv 2 \pmod{23}$ has no solution, since

By fermat's little theorem, $x^{23-1} \equiv 1 \pmod{23}$

$\therefore x^{22} \equiv 1 \pmod{23}$

Q.5. [TIFR'2011] T/F:- The value of the infinite product $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$ is 1.

Solution:- FALSE $\prod_{n=2}^{\infty}\left(1+\frac{1}{n}\right)\left(1-\frac{1}{n}\right)$

$$= \left(\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\right)\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\right)$$

$$= \frac{1}{2} \neq 1.$$

Q.6. [TIFR'2011] T/F:- $e^{\sqrt{2}} > 3$.

Sol. True. $e^{\sqrt{2}} > 4$.

$$e^{\sqrt{2}} = 1 + \frac{\sqrt{2}}{1!} + \frac{2}{2!} + \frac{2\sqrt{2}}{3!} + \cdots$$
$$> 3.$$

Q.7. [TIFR'2011] The equation $63x + 70y + 17z = 2010$ has an T/F:- integer solution.

Sol. True $(x, y, z) \equiv (15, 0, 71)$ is one integral solution of the equation.

Q.8. [TIFR'2012] T/F:- The inequality $\sum_{n=0}^{\infty} \frac{(\log\log 2)^n}{n!} > \frac{3}{5}$ holds.

Sol. TRUE

$$\sum_{n=0}^{\infty} \frac{(\log\log 2)^n}{n!} = e^{\log\log 2}$$
$$= \log 2$$

Now, $\log 2 > 3/5 \Rightarrow 5\log 2 > 3$

$\Rightarrow 2^5 > e^3$

This is true since $2 < e < 3$.

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Q.9. [TIFR'2012] The inequality $\sqrt{1+x} < 1+\frac{x}{2}$ for $x \in (-1,10)$ is true

Sol.

FALSE

AM $\geq$ GM

$$\frac{(1+x)+1}{2} \geq \sqrt{(1+x).1}$$

$$\Rightarrow 1+\frac{x}{2} \geq \sqrt{1+x}/2$$

'=' holds when $x=0$, so, for $x \in (-1,10)$, $\sqrt{1+x} < 1+\frac{x}{2}$ is not always true.

Q.10. [TIFR'2012] If n is not a multiple of 23 then the remainder when n^{11} is divided by 23 is $\pm 1 \pmod{23}$.

Sol.

TRUE

As n is not a multiple of 23 then by Fermat's little theorem:

$$n^{22} \equiv 1 \pmod{23}$$

$$\Rightarrow n^{11} \equiv \pm 1 \pmod{23}.$$

Hence, n^{11} is divided by 23 then remainder is ± 1.

Q.11. Which of the following statement is false?

(a) $\exists$ a natural number which when divided by 3 leaves remainder 1 and which when divided by 4 leaves remainder 0.

(b) $\exists$ a natural number which when divided by 6 leaves remainder 2 and when divided by 9 leaves remainder 1.

(c) $\exists$ a natural number which when divided by 7 leaves remainder 1 and when divided by 11 leaves remainder 3.

(d) $\exists$ a natural number which when divided by 12 leaves remainder 7 and when divided by 8 leaves remainder 3.

Solution:-

(b)

(a) $\longrightarrow$ Eg. 16

(c) $\longrightarrow$ e.g. 36

(d) $\longrightarrow$ e.g. 19

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[TIFR'2010]

Q.12. If n and m are positive integers and $n^9 = 19m + r$, then the possible values for r modulo 19 are

(a) only 0 (b) only $0, \pm 1$ (c) only ± 1 (d) none

Sol. (c)

$$n^9 \equiv r \pmod{19}$$

$$n^{18} \equiv r^2 \pmod{19}$$

By. fermat's little theorem, we know, $n^{18} \equiv 1 \pmod{19}$

So, $r^2 = 1 \Rightarrow r = \pm 1$.

Q. 13. [TIFR'2012] Given any integer $n \geq 2$, we can always find an integer m such that each of the $(n-1)$ consecutive integers $m+2, m+3, \ldots, m+n$ are composite.

Sol. True $n = 4$,

$m = 12$, then 14, 15, 16 are consecutive composite integers. So, the given statement is true.

Q. 14. [TIFR'2013] The inequality $\sqrt{n+1} - \sqrt{n} < \frac{1}{\sqrt{n}}$ is false for all n such that $101 \leq n \leq 2000$.

Sol. FALSE

$$(n+1)^{1/2} - n^{1/2}$$

$$= \left(n^{1/2} + \frac{1}{2}n + \cdots\right) - n^{1/2}, \text{ for large } n.$$

$$= \frac{1}{2n} < \frac{1}{n} < \frac{1}{\sqrt{n}} \text{ for all } n.$$

So, $\sqrt{n+1} - \sqrt{n} < \frac{1}{\sqrt{n}}$ is true for all n such that $101 \leq n \leq 2000$

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THEORY OF EQUATIONS

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Q.1. [TIFR'2010] Sum of the roots of polynomial equation $x^5 + 3x^2 + x + c = 0$ is 0. $\rightarrow$ T/F.

Solution:- True

$$\text{Sum of all roots} = \left(-\frac{\text{coefficient of } x^4}{\text{coefficient of } x^5}\right) = -\left(\frac{0}{1}\right) = 0.$$

Q.2. [TIFR'2011] The sum of the squares of the roots of the cubic equation $x^4 - 4x^2 + 6x + 1$ is

(a) 0 (b) 4 (c) 16 (d) None

Solution:- (b)

Let the 3 roots be x_1, x_2, x_3.

Now, $x_1 + x_2 + x_3 = -\frac{(-4)}{1} = 4.$

$$x_1x_2 + x_1x_3 + x_2x_3 = 6$$

$$x_1x_2x_3 = -1.$$

So, now, $(x_1 + x_2 + x_3)^2 = 16$

$$\Rightarrow x_1^2 + x_2^2 + x_3^2 + 2(x_1x_2 + x_1x_3 + x_2x_3) = 16$$

$$\Rightarrow x_1^2 + x_2^2 + x_3^2 = 16 - 12 = 4.$$

Q.3. [TIFR'2011] T/F:- For any real number c, the polynomial $x^3 + x + c$ has exactly one real root.

Solution:- True The polynomial has exactly one real root if $c = 2$.

$$x^3 + x + 2 = (x+1)(x^2 - x + 2)$$

$\therefore x = -1$ is the real root.

Q.4. [TIFR'2011] T/F:- The polynomial $x^4 + 7x^3 - 13x^2 + 11x$ has exactly one real root.

Solution:- False The given polynomial is of even degree that is it can have at most 4 roots of which it can either have 4 or 2 or zero real roots since complex roots occur in pairs. But it can't have exactly one real root.

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Q.5. [TIFR '2011] T/F:- Let P be a degree 3 polynomial with complex coefficients & the constant term is 2010. Then P has a root α with $|\alpha| > 10$.

Solution:- False

Let $P(X) = AX^3 + BX^2 + CX + D$

Since $P(X)$ is cubic so one of its root must be real and rest of two are conjugate of each other, here A, B, C, D are complex numbers. But A and D must be real here $D = 2010$.

If the roots are α, α, β.

$$\text{So, } \alpha^2\beta = -\frac{2010}{A}.$$

If $A = 67$, $B = 5$, then $|\alpha| < 10$.

So, when $A \geq 67$ then the statement is false.

Q.6. [TIFR '2012] T/F:- There exists polynomials $f(x)$ and $g(x)$ with complex coefficients, such that $\left(\frac{f(x)}{g(x)}\right)^2 = x$.

Solution:- False

Counter Example:- $f(x) = ix^2 + 3$, $g(x) = i^2x^2 - 5$, then

$$\left(\frac{f(x)}{g(x)}\right)^2 \neq x.$$

Q.7. [TIFR '2013] The equation $x^3 + 3x - 4 = 0$ has exactly one real root.

Solution:- $(x-1)(x^2+x+4) = 0$

TRUE

$$\Rightarrow x = 1, \quad x = \frac{-1 \pm \sqrt{1-16}}{2} = \frac{-1 \pm \sqrt{-15}}{2}.$$

So, the equation has exactly one real root.

Q.8. [TIFR '2013] The equation $x^3 + 10x^2 - 100x + 1729 = 0$ has at least one complex root α such that $|\alpha| > 12$.

Solution:- TRUE

Apply the above method shown in Q.5. [TIFR '11].

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ANALYSIS

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Q.1. [TIFR'2010] The sequence $\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \ldots\ldots$ converges to

(a) $\frac{1+\sqrt{33}}{2}$ (b) $\frac{1+\sqrt{32}}{2}$ (c) $\frac{1+\sqrt{30}}{2}$ (d) $\frac{1+\sqrt{29}}{2}$

Solution:- (d) The sequence is of the form:

$$l = \sqrt{7+l}$$

$$\Rightarrow l^2 - 7 - l = 0$$

$$\Rightarrow l = \frac{1 \pm \sqrt{1+28}}{2}$$

So, the sequence converges to $\frac{1+\sqrt{29}}{2}$ since there can't be negative root.

Note:- If $a_{n+1} = \sqrt{K + a_n}$, then $\{a_n\}$ converges to the positive root of the equation $x^2 - x - K = 0$.

Q.2. [TIFR'2011] T/F:- The derivative of the function $\int_0^{\sqrt{x}} e^{-t^2} dt$ is e^{-1} at $x=1$.

Solution:- Leibnitz Rule:- If $f(x)$ is continuous and $u(x)$ and $v(x)$ are differentiable functions in the interval $[a,b]$ then

FALSE $$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f\{v(x)\}\frac{d}{dx}(v(x)) - f\{u(x)\}\frac{d}{dx}(u(x)).$$

Using Leibnitz Rule, $$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = \frac{d}{dx}\int_0^{\sqrt{x}} e^{-t^2} dt$$

$$= e^{-x}\cdot\frac{1}{2\sqrt{x}} - 1\cdot\frac{d}{dx}(1)$$

$$= e^{-x}\cdot\frac{1}{2\sqrt{x}}$$

At $x=1$, Ans. is $= \frac{e^{-1}}{2} \neq e^{-1}$.

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Q.3. [TIFR'2012] T/F:- $f:[0,\infty]\to[0,\infty]$ is continuous and bounded then f has a fixed point.

Sol. TRUE Then $f(x)-x\neq 0\ \forall\ x\in[0,\infty]$.

Then $f(x)-x$ must be positive for some α in $[0,\infty]$

and $f(x)-x$ must be negative for some β in $[0,\infty]$

Then f is continuous and $f(x)-x$ assumes $-ve$ to $+ve$ values from α to β in $[0,\infty]$ and therefore by Intermediate value theorem $\exists$ a point c lying between α and β in $[0,\infty]\ni f(c)=c\ \forall\ c\in[0,\infty]$.

$\therefore$ f has a fixed point.

[Intermediate Value Theorem:- Let $[a,b]$ be a closed and bounded interval and a function $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. If $f(a)\neq f(b)$ then f attains every value between $f(a)$ and $f(b)$ at least once in the open interval (a,b).]

Q.4. [TIFR'2012] T/F:- The functions $f(x)=x|x|$ and $g(x)=x|\sin x|$ are not differentiable at $x=0$.

Solution:- FALSE $f'(x)=|x|+x$ $g'(x)=x|\cos x|+|\sin x|$.

Both differentiable at $x=0$.

Q.5. [TIFR'2012] T/F:- Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{h}$$ exists for all $x\in\mathbb{R}$, Then f is differentiable in $\mathbb{R}$.

Sol. FALSE $f(x)=|x|$,

$$\lim_{h\to 0}\frac{f(h)-f(-h)}{h}=0$$ exists but $f'(0)$ does not exist.

This is an counter example of the above statement.

Note:- A function f is defined on some neighbourhood of c and f is differentiable at c, then

$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c-h)}{2h}.$$

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6. [TIFR'2012] T/F:- The composition of two uniformly continuous functions needs not always be uniformly continuous.

Sol. TRUE Let $f(x) = x^2$ is uniformly continuous on $[-1,1]$ and $g(x) = \sin x^2$ is uniformly continuous on $[-1,1]$.

But $h(x) = x^2 \sin x^2$ is not uniformly continuous on $[-1,1]$.

Let $\epsilon = 1/2$, $\delta > 0$ such that for $x > \frac{\pi}{\delta^2}$,

let, $x_1 = \sqrt{\frac{x\pi}{2}}$, $x_2 = \sqrt{\frac{(x+1)\pi}{2}}$ as any two points of the interval $[-1,1]$, then

$$\left| \sqrt{\frac{x\pi}{2}} - \sqrt{\frac{(x+1)\pi}{2}} \right| < \delta.$$

Now, $$|h(x_1) - h(x_2)| = \left| \left(\frac{x\pi}{2}\right) \sin\left(\frac{x\pi}{2}\right) - \left(\frac{x+1}{2}\right)\pi \sin\left(\frac{x+1}{2}\right)\pi \right| > \epsilon$$

and $|x_1 - x_2| < \delta$.

Hence, $x^2 \sin x^2$ is not uniformly continuous on $[-1,1]$.

7. [TIFR'2012] T/F:- Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then the derivative $\frac{\partial^2 f}{\partial x \partial y}$ can exist without $\frac{\partial f}{\partial x}$ existing.

Sol. TRUE If $\frac{\partial f}{\partial y}$ exists then $\frac{\partial^2 f}{\partial x \partial y}$ can exist.

8. [TIFR'2012] T/F:- The integral $\int_{-\infty}^{\infty} \frac{e^{-x}}{1+x^2}\, dx$ is convergent.

Sol. TRUE

$$\left| \frac{e^{-x}}{1+x^2} \right| \leq \frac{1}{1+x^2}$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \lim_{\substack{x \to -\infty \\ y \to \infty}} \int_{x}^{y} \frac{1}{1+x^2}\, dx = \lim_{\substack{x \to -\infty \\ y \to \infty}} \left[\tan^{-1} y - \tan^{-1} x\right]$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

$$= \pi.$$

$\therefore \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is convergent; by Comparison test, $\int_{-\infty}^{\infty} \frac{e^{-x}}{1+x^2}\, dx$ converges.

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9. [TIFR'2010] Let $u_n = \sin\left(\frac{\pi}{n}\right)$ and consider the series $\sum u_n$. Which of the following is TRUE ?

(a) $\sum u_n$ is convergent
(b) $u_n \to 0$ as $n \to \infty$.
(c) $\sum u_n$ is divergent
(d) $\sum u_n$ is absolutely convergent.

Sol. (c)

$$u_n = \sin\frac{\pi}{n} = \frac{\pi}{n} - \frac{1}{3!}\left(\frac{\pi}{n}\right)^3 + \cdots\cdots \infty$$

$$v_n = \frac{1}{n}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \pi \neq 0$$

As, $\sum v_n$ is convergent for $p > 1$.
Here $p = 1$. So, $\sum v_n$ diverges, So does $\sum u_n$.

10. [TIFR'2010] If $f, g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous functions, then their composition $g \circ f$ is

(a) uniformly continuous
(b) continuous but not uniformly continuous
(c) continuous and bounded
(d) None of the above.

Sol. (a)

Since functions f and g are continuous then their composite function $g \circ f$ is also continuous but we can't say anything about its uniform continuity.
But here f, g are uniformly continuous, then $g \circ f$ is uniformly continuous.

11. [TIFR'2010, 2011] T/F:- The function $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$ is not continuous anywhere on the real line.

Solution:- TRUE We know that there exists infinitely many rational numbers between two irrational numbers and similarly there may be infinitely many irrationals in the neighbourhood of a rational.
So, $f(x)$ is not continuous anywhere on the real line.

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12. [TIFR'2011] Any continuous function from the open unit interval $(0,1)$ to itself has a fixed point.

T/F:-

Sol. FALSE

Counter Example:- Let $f(x) = x^2$.

For $c \in (0,1)$,

$$c - f(c) = c - c^2 = c(1-c) < c.$$

Since $0 < 1-c < 1$. Hence, $f(c) \neq c$ for all $c \in (0,1)$, so f has no fixed points.

13. [TIFR'2011] Consider the sequence $\{x_n\}$ defined by $x_n = \frac{[nx]}{n}$ for $x \in \mathbb{R}$, where $[\cdot]$ denotes the integer part. Then $\{x_n\}$

(a) converges to x.
(b) converges but not to x
(c) does not converge
(d) oscillates.

Sol. (a) $x_n = \frac{[nx]}{n} \leq \frac{nx}{n} = x.$

$$\therefore \lim_{n\to\infty} \{x_n\} = \lim_{n\to\infty} (x) = x.$$

14. [TIFR'2011] $\lim_{x\to 0} x\sin\left(\frac{1}{x^2}\right)$ equals

(a) 1 (b) 0 (c) ∞ (d) oscillates

Sol. (b)

Here $\lim_{x\to 0} x = 0$ and $\sin\left(\frac{1}{x^2}\right)$ is bounded in some deleted neighbourhood of 0. Therefore $\lim_{x\to 0} x\sin\frac{1}{x^2} = 0$.

15. [TIFR'2011] T/F:- The function $f_n(x) = x\sin\left(\frac{x}{n}\right)$ converges to the function x as $n \to \infty$.

Sol. TRUE

$$\lim_{n\to\infty} x\sin\left(\frac{x}{n}\right)$$

$$-1 \leq \sin\left(\frac{x}{n}\right) \leq 1$$

$$-x \leq x\sin\left(\frac{x}{n}\right) \leq x$$

$$\therefore \lim_{n\to\infty} x\sin\left(\frac{x}{n}\right) = x.$$

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16. [TIFR'2011] The function $f(x)$ defined by $f(x) = \begin{cases} ax+b, & x \geq 1 \\ x^2+3x+3, & x \leq 1 \end{cases}$

is differentiable

(a) for a unique value of a and infinitely many values of b.

(b) for a unique value of b and infinitely many values of a.

(c) for infinitely many values of a and b.

(d) none.

<u>Solution:-</u> (d) Clearly, the function $f(x)$ is differentiable at all the values of x except $x=1$.

So, we will check the differentiability at $x=1$.

$$Rf'(1) = \lim_{h\to 1+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 1+} \frac{a(1+h)+b-7}{h} = \lim_{h\to 1+} a + \lim_{h\to 1+} \frac{a+b-7}{h}$$

$$= a.$$

Since $f(x)$ is differentiable then $Rf'(1)$ has finite and unique quantity

So, $a+b-7=0 \Rightarrow a+b=7$.

$$Lf'(1) = \lim_{h\to 1-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 1-} \frac{(1+h)^2+3(1+h)+3-7}{h} = \lim_{h\to 1-} (h+5)$$

$$= 5.$$

As, differentiability implies $Lf'(1) = Rf'(1)$

so, $a=5$ and hence $b=2$.

Hence we get unique values of both a and b.

17. [TIFR'11] <u>T/F:-</u> $\log x$ is uniformly continuous on $\left(\frac{1}{2}, \infty\right)$.

<u>Sol.</u> FALSE We take two points $x, y \in \left(\frac{1}{2}, \infty\right)$ and $\epsilon > 0$.

Now, $|\log x - \log y| = \left|\log \frac{x}{y}\right| < \epsilon$

$$\Rightarrow \left|\frac{x}{y}\right| < e^{\epsilon}$$

$$\Rightarrow |x| < e^{\epsilon}|y|.$$

Taking $e^{\epsilon} = \delta$, $\Rightarrow |x| < \delta|y|$

$\therefore |x-y| = (1-\delta)x$, is not a small quantity.

So, $\log x$ is not uniformly continuous.

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18. [TIFR'2011] T/F:- The series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$ diverges.

Solution:- FALSE

$$u_n = \sqrt{\frac{1}{n}+\frac{1}{n^2}} - \sqrt{\frac{1}{n}}$$

$$= \frac{1}{\sqrt{n}}\left(1+\frac{1}{2n}+\cdots-1\right)$$

$$= \frac{1}{2n^{3/2}} + O\left(\frac{1}{n^{3/2}}\right)$$

Take $v_n = \frac{1}{n^{3/2}}$.

So, since $\sum v_n$ converges, so does $\sum u_n$.

Alter:-

$$u_n = \frac{1}{n(\sqrt{n+1}+\sqrt{n})}$$

$$\frac{1}{2n^{3/2}} < \frac{1}{2n(\sqrt{n+1})} < \frac{1}{n(\sqrt{n+1}+\sqrt{n})}$$

As, $\sum \frac{1}{2n^{3/2}}$ converges so does $\sum u_n$, by Comparison test.

19. [TIFR'2011] T/F:- Suppose $f_n(x)$ is a sequence of continuous functions on the closed interval $[0,1]$ converging to 0 pointwise. Then the integral $\int_0^1 f_n(x)\,dx$ converges to 0.

Sol. TRUE $\lim_{n\to\infty} f_n(x) = 0 \quad (0 \le x \le 1)$

for $f_n(0) = 0$; $n \in I$.

While if $x > 0$ then $f_N(x) = f_{N+1}(x) = \cdots = 0$.

if $2/N < x$

Hence, $\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = 0 = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx$ for $2/N < x$.

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20. [TIFR'2011] T/F :- Let f be a continuous integrable function on $\mathbb{R}$ such that either $f(x) > 0$ or $f(x) + f(x-1) > 0$ $\forall x \in \mathbb{R}$. Then

$$\int_{-\infty}^{\infty} f(x)\,dx \geq 0$$

Sol. TRUE Let f be a continuous integrable function on $\mathbb{R} \ni f(x) > 0$

Then $\int_{-\infty}^{\infty} f(x)\,dx \geq 0$

As, by the inequality on bounded integrable function f

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a), \text{ when } b \geq a.$$

So, if $f(x) \geq 0$ then lower bound $m \geq 0 \Rightarrow \int_{-\infty}^{\infty} f(x)\,dx \geq 0$.

21. [TIFR'2010] $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$ is

(a) 1 (b) 0 (c) $1/2$ (d) does not exist.

Solution:- (b) $\lim_{x \to 0} \frac{\sin(1/x)}{(1/x)} = \lim_{z \to \infty} \frac{\sin z}{z} = 0$.

22. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

(a) Converges but not absolutely (b) converges absolutely
(c) diverges (d) None of the above

sol. (a) $\sqrt{n+1} > \sqrt{n}$

$u_n = n^{-1/2} > (n+1)^{-1/2}$

$\sum (-1)^{n+1} u_n$; $\{u_n\}$ is monotone decreasing

$\therefore \lim u_n = 0$.

$\therefore$ By Leibnitz test the series converges.

Now, $|u_n| = \frac{1}{n^{1/2}}$. By Comparison test $\sum \frac{1}{\sqrt{n}}$ is convergent,

So, the series is not absolutely convergent.

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Q.23. [TIFR'2013] T/F:- Consider the sequence $x_n = \sum_{j=1}^{n} \frac{1}{j}$, $y_n = \sum_{j=1}^{n} \frac{1}{j^2}$. Then $\{x_n\}$ is cauchy but $\{y_n\}$ is not.

Sol. :→ FALSE

$$\sum u_n = \sum_{j=1}^{n} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$|u_{n+p} - u_n| = \frac{1}{n+1} + \cdots + \frac{1}{n+p}.$$

Choose, $n = m$, $p = m$.

Then $|u_{2m} - u_m| = \frac{1}{m+1} + \cdots + \frac{1}{2m}$

$$> \frac{1}{2m} + \cdots + \frac{1}{2m} = \frac{1}{2}.$$

If we choose $\epsilon = 1/2$ then no natural number k can be found $\ni$ $|u_{n+p} - u_n| < \epsilon$ coill hold $\forall$ $n \geq k$ and every natural number k.

So, $\{u_n\}$ is not convergent.

So, $\{u_n\}$ is not cauchy.

i.e. $\{x_n\}$ is not cauchy.

Q.24. [TIFR'2013] T/F:- $\lim_{x \to 0} \frac{\sin(x^2)}{x^2} \cdot \sin\left(\frac{1}{x}\right) = 1$.

Sol. FALSE

$$\lim_{x \to 0} \frac{\sin(x^2)}{x^2} \cdot \sin\left(\frac{1}{x}\right) = \lim_{x \to 0} \frac{x^2}{x^2} \cdot \sin\left(\frac{1}{x}\right)$$

$$= \lim_{x \to 0} \sin\frac{1}{x}, \text{ it does not exist.}$$

Q.25. [TIFR'2013] T/F:- Let $f: [a,b] \to [c,d]$ and $g: [c,d] \to \mathbb{R}$ be Riemann integrable functions defined on the closed intervals $[a,b]$ and $[c,d]$ respectively. Then the composite $g \circ f$ is also Riemann integrable.

Sol. FALSE

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Q.26. [TIFR'2013] T/F:- Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin x^3$. Then f is continuous but not uniformly continuous.

Sol. TRUE

Q.27. [TIFR'2013] T/F:- Suppose $\{a_i\}$ is a sequence in $\mathbb{R}$ such that $\sum |a_i||x_i| < \infty$ cohenever $\sum |x_i| < \infty$. Then $\{a_i\}$ is a bounded sequence.

Sol. TRUE

Q.28. [TIFR'2013] T/F:- The integral $\int_0^\infty e^{-x^5} dx$ is convergent.

Sol. TRUE

$$I = \int_0^\infty e^{-x^4} dx = \frac{1}{4} \int_0^\infty e^{-y} y^{1/4 - 1} dy, \quad y = x^4$$

We know Gamma integral converges iff $n > 0$,

$$\Gamma(n) = \int_0^\infty e^{-y} y^{n-1} dy.$$

Here $n = 1/4$.

So, $I = \frac{1}{4} \Gamma\left(\frac{1}{4}\right) = \Gamma\left(\frac{5}{4}\right)$.

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Q.29. [TIFR'2013] T/F:- Let $P(x) = 1 + x + x^2/2! + x^3/3! + \cdots + x^n/n!$ where n is a large positive integer. Then $\lim_{x\to\infty} \frac{e^x}{P(x)} = 1$.

Sol. FALSE

Apply L'Hospital Rule repeatedly.

$$\lim_{x\to\infty} \frac{e^x}{P(x)} \equiv \lim_{x\to\infty} \frac{e^x}{\text{constant}} = \infty.$$

Q.30. [TIFR'2013] T/F:- Every differentiable function $f:(0,1)\to[0,1]$ is uniformly continuous.

Sol. FALSE Let $f(x) = x^2 \sin\frac{1}{x^2}$ is from $(0,1)$ to $[0,1]$ whose derivative is unbounded.

Here $f(x)$ is not uniformly continuous.

Q.31. [TIFR'2013] T/F:- Let f be a function defined on the closed interval $[0,1]$ defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$$

Then f is continuous at 0 and 1.

Sol. TRUE

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Q.32. [TIFR'2013] T/F:- $\lim_{n\to\infty} (n+1)^{1/3} - n^{1/3} = \infty$.

Sol. FALSE

$$\lim_{n\to\infty}\left\{(n+1)^{1/3} - n^{1/3}\right\}$$

$$= \lim_{n\to\infty}\left[\left\{n^{1/3} + \frac{1}{3}n^{-2/3} + \cdots\right\} - n^{1/3}\right]$$

$$= \lim_{n\to\infty} \frac{1}{3n^{2/3}}$$

$$= 0.$$

Q.33. [TIFR'2013] T/F:- Let $\{a_n\}$ be any non-constant sequence in $\mathbb{R}$ such that $a_{n+1} = \frac{a_n + a_{n+2}}{2}$ $\forall\, n \geq 1$. Then $\{a_n\}$ is unbounded.

Sol. TRUE

$$a_2 = \frac{a_1 + a_3}{2}$$

$$a_3 - a_2 = a_2 - a_1$$

$$a_4 - a_3 = a_3 - a_2$$

$$\vdots$$

$$a_{n+2} - a_{n+1} = a_{n+1} - a_n$$

$a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}$

Terms are in A.P., can be in increasing or decreasing order of sequence.

So, it is unbounded.

Q.34. [TIFR'2013] T/F:- The series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ is divergent.

Sol. FALSE

Let $a_n = \frac{1}{n^p}$, $n \in \mathbb{N}$

$$a_{n+1} - a_n = \frac{1}{(n+1)^p} - \frac{1}{n^p} < 0 \quad \forall\, n \in \mathbb{N}$$

$\Rightarrow \{a_n\}$ is decreasing.

Note that, $\lim (a_n) = \lim \left(\frac{1}{n^p}\right) = 0$ if $p > 0$.

By Leibnitz test, the alternating series $\sum \frac{(-1)^{n-1}}{n^p}$ converges if $p > 0$.

Here $p = 1/2$. So, the series is convergent.

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Q.35. [TIFR'2014] Let A, B, C be three subsets of $\mathbb{R}$. The negation of the following statement

For every $\epsilon > 1$, there exists $a \in A$ and $b \in B$ such that for all $c \in C$, $|a-c| < \epsilon$ and $|b-c| < \epsilon$ is

A. $\exists\ \epsilon \leq 1$, such that for all $a \in A$ and $b \in B$ $\exists\ c \in C$ such that $|a-c| \geq \epsilon$ or $|b-c| \leq \epsilon$.

B. $\exists\ \epsilon \leq 1$, such that for all $a \in A$ and $b \in B$ $\exists\ c \in C \ni$ $|a-c| \geq \epsilon$ and $|b-c| \leq \epsilon$.

C. $\exists\ \epsilon > 1$, such that for all $a \in A$ and $b \in B$ $\exists\ c \in C \ni$ $|a-c| \geq \epsilon$ and $|b-c| \leq \epsilon$

D. $\exists\ \epsilon > 1$, such that for all $a \in A$ and $b \in B$ $\exists\ c \in C \ni$ $|a-c| \geq \epsilon$ or $|b-c| \leq \epsilon$.

Solution:- (D)

[TIFR'2014]

Q.38. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous bounded function, then:

A. f has to be uniformly continuous

B. there exists an $x \in \mathbb{R}$ such that $f(x) = x$

C. f cannot be increasing

D. $\lim_{x \to \infty} f(x)$ exists.

Sol. (B)

$f(x) = \sin(x^3)$ is a continuous function which is bounded by 1 which is not uniformly continuous.

$f(x) = \arctan(x)$ / $\tan^{-1} x$.

$f(x) = \sin(x^3)$ has no limit at infinity.

Take $g(x) = x - f(x)$, and as f is bounded so g having both signs, ($g(x) \to \inf$ and $-\inf$ as $x \to \inf$ and $-\inf$), i.e. for some a, b belongs to $\mathbb{R}$ s.t. $g(a) < 0$, $g(b) > 0$

So, by Bolzano/ IVP, there is a c in $\mathbb{R}$ s.t. $g(c) = 0$, i.e. $f(c) = c$

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[TIFR'2014]

Q.37. Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $\lim_{x \to +\infty} f'(x) = 1$, then

A. f is bounded B. f is increasing C. f is unbounded

D. f' is bounded.

Sol. (c)

Let $f(x) = x$, then f is not bounded.

Consider another function $f(x) = \begin{cases} \arctan x + x, & \text{if } x \geq 0 \\ x - x^2, & \text{if } x \leq 0 \end{cases}$

$$\lim_{x \to \infty} \left(\frac{1}{1+x^2} + 1 \right) = 1.$$

Also its non-increasing on $(-1, -1/2)$.

Also, let us see why f has to be unbounded.

Fix a sufficiently large so that for any $c > a$, $f'(c) \in [1/2, 3/2]$. We can do this because $\lim_{x \to \infty} f'(x) = 1$. Then for any $b > a$, the MVT shows that there is $d \in [a, b]$ so that

$$\frac{f(b) - f(a)}{b - a} = f'(d).$$

Now, the right hand side is at least $1/2$ and so $f(b) \geq 0.5(b-a) + f(a)$. Let b tend to infinity and we see that f is unbounded.

Q.38. [TIFR'2014] Let $f_n(x)$, for $n \geq 1$, be a sequence of continuous non-negative functions on $[0,1]$ such that

$$\lim_{n \to \infty} \int_0^1 f_n(x)\, dx = 0$$

Which of the following statement is always correct?

A. $f_n \to 0$ uniformly on $[0,1]$

B. f_n may not converge uniformly but converges to 0 pointwise

C. f_n will converge pointwise and the limit may be non-zero

d. f_n is not guaranteed to have a point-wise limit.

Sol. Take $f_n(x) = x^n$.

It is not uniformly convergent to 0 on $[0,1]$ but the limiting value is 0.

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LINEAR ALGEBRA

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Q.1. [TIFR'2010] Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices with real entries. Which of the following statement is true?

(a) Any matrix $A \in M_4(\mathbb{R})$ has a real eigen value.
(b) Any matrix $A \in M_5(\mathbb{R})$ has a real eigenvalue.
(c) Any matrix $A \in M_2(\mathbb{R})$ has a real eigenvalue.
(d) none of the above.

Sol. (b) Complex roots occur in pairs.
As $M_5(\mathbb{R})$ has 5 eigen values of which one must be real.

Q.2. [TIFR'2010] Which of the following is TRUE?

(a) The matrix $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ is not diagonalisable.
(b) The matrix $\begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}$ is diagonalisable.
(c) The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is diagonalisable.
(d) None of the above.

Sol. (b) $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 5 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 1, 2.$$

Since the 2×2 matrix has 2 distinct eigenvalues so it is diagonalisable.

Q.3. If V is a vector space over the field $\mathbb{Z}/5\mathbb{Z}$ and $\dim_{\mathbb{Z}/5\mathbb{Z}}(V) = 3$, then V has

(a) 125 elements (b) 15 elements (c) 243 elements (d) none.

Sol. (a)

$$F = \mathbb{Z}_5$$
$$\dim(V) = 3$$

So, it has 5^3 elements.

Remark:- This question can be alternatively asked as ←

Write T/F :- If a vector space $V = \{(a,b,c) \mid a,b,c \in \mathbb{Z}_5\}$, then it has 125 elements.

ANS:- TRUE

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Q.4. [TIFR'2010] The equations
$$x_1 + 2x_2 + 3x_3 = 1$$
$$x_1 + 4x_2 + 9x_3 = 1$$
$$x_1 + 8x_2 + 27x_3 = 1$$
have

(a) Only one solution
(b) two solutions
(c) infinitely many solutions
(d) No solution.

Sol. (a) [$x_1 = 1, x_2 = 0, x_3 = 0$ is the only solution]

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{bmatrix}, \quad \text{so} \quad |A| \neq 0 \qquad \underset{\sim}{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So, $\text{rank}(A) = 3$, $\text{rank}(A \mid b) = 3$.

So, the system of linear equations has only one solution.

Q.5. Let $\underset{\sim}{x}$ and $\underset{\sim}{y}$ in $\mathbb{R}^n$ be non-zero column vectors. From the matrix $A = xy^T$, where y^T is the transpose of y. Then the rank(A) is [TIFR'10]

(a) 2
(b) 0
(c) at least $n/2$
(d) None of the above.

Sol. (d)
$$\underset{\sim}{x} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}_{m\times 1}, \quad \underset{\sim}{y} = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix}_{n\times 1} \quad (\text{WLG take } m<n)$$

$$A_{m\times n} = xy^T$$
$$= \begin{bmatrix} a_{11}b_{11} & \cdots & a_{11}b_{n1} \\ \vdots & & \vdots \\ a_{m1}b_{11} & \cdots & a_{m1}b_{n1} \end{bmatrix}_{m\times n}$$

So, $A_{m\times n}$ will be a non-zero matrix. Since the matrix will have at least one non-zero element obtained by multiplying corresponding non-zero elements of $\underset{\sim}{x}$ and y^T. So, All the two sized minors of AB clearly vanish.

$\therefore$ Rank (A) = 1.

Remark:- If A be a non-zero column matrix of order $m\times 1$ and B be a non-zero matrix of order $1\times n$, then $\text{rank}(AB) = 1$.

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Q.6. [TIFR'2011] Let A be a 5X5 matrix with real entries, then A has
(a) an eigen value which is purely imaginary.
(b) at least one real eigen value.
(c) At least two eigenvalue which are not real.
(d) at least two distinct real eigen values.

Sol. (b) A has 5 eigen values of which at least one must be real since complex roots occur in pairs.

Q.7. [TIFR'2011] T/F:- A is 3X4 matrix of rank 3. Then the system of equations $Ax = b$ has exactly one solution.

Sol. FALSE

$r(A) = 3$

$r(A|b) = 3$

$\Rightarrow$ System of linear equation is consistent.

It has 4 variables in 3 equations.

i.e. it will have infinite number of solutions.

Q.8. [TIFR'2011] T/F:- Consider the map T from the vector space of polynomials of degree at most 5 over the reals to $\mathbb{R} \times \mathbb{R}$, given by sending a polynomial P to the pair $(P(3), P'(3))$, where $P'(3)$ is the derivative of P. Then the dimension of the kernel is 3.

Sol. FALSE Since $T: P_5[X] \to \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

Here $\dim P_5[X] = 6$, $\dim \mathbb{R}^2 = 2$.

By Sylvester's law of nullity:

$\dim P_5[X] = \text{Rank} + \text{Nullity}$

$\Rightarrow 6 = 2 + \text{Nullity}$

$\Rightarrow \text{Nullity} = 4.$

Q.9. [TIFR'2011] T/F:- If A and B are 3X3 matrices and A is invertible, then $\exists$ an integer n such that $A + nB$ is invertible.

Sol. TRUE $|A| \neq 0$; (i) $|B| = 0$ both implying that
(ii) $|B| \neq 0$ $|A + nB| \neq 0 \ \forall n \in I$.

Then $(A + nB)^{-1}$ exists.

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Q.10. [TIFR'11] T/F:- Let A be a 2X2 matrix with complex entries. The number of 2X2 matrices A with complex entries satisfying the equation $A^3 = A$ is infinite.

Sol. TRUE If A be a 2X2 matrix with complex entries such that

$$A^3 = A$$

$$\Rightarrow A^3 - A = 0 \Rightarrow A(A^2 - I) = 0 \Rightarrow A = 0_{2\times2},\ A^2 = I_{2\times2}$$

There are infinitely many 2X2 matrices A with complex entries satisfying $A^3 = A$ or $A^2 = I$.

Q.11. [TIFR'11] T/F:- Let S be a finite subset of $\mathbb{R}^3$ such that any three elements in S span a two dimensional subspace. Then S spans a two dimensional space.

Sol. TRUE Let S be a finite subset of $\mathbb{R}^3$ $\ni$ any three elements in S span a two dimensional subspace.

Since the dimension of a two dimensional space is two.

$\Rightarrow$ any set having two elements or more can span S.

$\Rightarrow$ A three element set from $\mathbb{R}$ can also span $\mathbb{R}^2$.

Q.12. [TIFR'11] T/F:- Any non-singular KXK matrix with real entries can be made singular by changing exactly one entry.

Sol. FALSE

Some of matrices could be converted to singular by changing exactly one entry but this could not be true with any matrix.

Like $A = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$ is a non-singular matrix, if we replace a_{11} by 6. Then $B = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$ is a singular matrix.

But in general, it does not happen with any matrix.

Q.13. [TIFR'2012] T/F:- The matrix $\begin{pmatrix} 1 & \pi & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{pmatrix}$ is diagonalisable.

Sol. TRUE $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & \pi & 3 \\ 0 & 2-\lambda & 4 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\Rightarrow \lambda = 1, 2, 3.$$

Since the matrix A has 3 distinct eigen values, so, the matrix is diagonalisable.

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Q.14. [TIFR '2012] T/F :- Suppose A is a nilpotent matrix and I is the identity matrix. Then $(I+A)$ is invertible.

Sol. TRUE

$$B = (A+I) = (I+A)$$
$$(A+I)(I-A)^{-1} = I$$
$$\Rightarrow BB^{-1} = I$$
$$\Rightarrow B \text{ is invertible.}$$

Q.15. [TIFR '2012] T/F :- The equation

$$x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 = 1$$
$$x_1 + \frac{1}{4}x_2 + \frac{1}{9}x_3 = 1$$
$$x_1 + \frac{1}{8}x_2 + \frac{1}{27}x_3 = 1$$

has no solution.

Sol. FALSE

$x_1 = 1, x_2 = 0, x_3 = 0$ is one of the solution of the system of equation.

Q.16. [TIFR '12] T/F :- Any odd order skew symmetric matrices have always zero determinants.

Sol. TRUE

$$\det \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = 0.$$

Q.17. [TIFR '12] T/F :- The rank of the matrix $\begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$ is 2.

Sol. TRUE

Perform $R_{i+1} = R_{i+1} - R_i \ \forall\, i = 1, 2, 3.$

We get, $\begin{bmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

Rank is $= 2$.

Q.18. [TIFR '12] T/F :- The 10×10 matrix $\begin{pmatrix} v_1 w_1 & \cdots & v_1 w_{10} \\ v_2 w_1 & \cdots & v_2 w_{10} \\ \vdots & & \vdots \\ v_{10} w_1 & \cdots & v_{10} w_{10} \end{pmatrix}$ has rank 2.

Sol. FALSE

$$V_{10\times 1} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{10} \end{pmatrix}, \quad W_{1\times 10} = (w_1 \cdots w_{10})$$

$\therefore$ Rank $(VW) = 1 \neq 2$. [See Q.5.]

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Q.19. [TIFR'2012] The polynomials $(t-1)(t-2)$, $(t-2)(t-3)$, $(t-3)(t-4)$, $(t-4)(t-6) \in \mathbb{R}[t]$ are linearly independent.

Sol. FALSE If the polynomials are linearly independent then

let
$$\alpha_1 = t^2 - 3t + 2$$
$$\alpha_2 = t^2 - 5t + 6$$
$$\alpha_3 = t^2 - 7t + 12$$
$$\alpha_4 = t^2 - 10t + 24$$

and let $\lambda_i \in \mathbb{R}$ be the scalars $\forall i = 1(1)4$.

Then for LIN, $\lambda_i = 0$ is the only solution of the equation

$$\lambda_1 \underset{\sim}{\alpha_1} + \lambda_2 \underset{\sim}{\alpha_2} + \lambda_3 \underset{\sim}{\alpha_3} + \lambda_4 \underset{\sim}{\alpha_4} = \underset{\sim}{0}$$

$$\Rightarrow \lambda_1 (t^2 - 3t + 2) + \lambda_2 (t^2 - 5t + 6) + \lambda_3 (t^2 - 7t + 12) + \lambda_4 (t^2 - 10t + 24) = \underset{\sim}{0}$$

$$\Rightarrow (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) t^2 + (-3\lambda_1 - 5\lambda_2 + 7\lambda_3 - 10\lambda_4) t + (2\lambda_1 + 6\lambda_2 + 12\lambda_3 + 24\lambda_4) = (0, 0, 0)$$

On solving, we get $\lambda_3 = -\frac{4}{15}\lambda_4$.

$\therefore$ These polynomials are not linearly independent.

Q.20. [TIFR'2012] $A \in M_2(\mathbb{C})$ and A is nilpotent then $A^2 = 0$.

Sol. TRUE $A = \begin{bmatrix} 1 & -i^2 \\ 1 & -i^2 \end{bmatrix}$, then A is nilpotent and $A \in M_2(\mathbb{C})$.

And if we check we will get $A^2 = 0_{2\times 2}$.

Q.21. [TIFR'2012] Let P be an $n\times n$ matrix whose row sums equal 1. Then for any positive integer m the row sums of the matrix P^m equal 1.

Sol. TRUE

Take any example. Take $P_{2\times 2} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, row sums equal 1.

Let $m = 2$, then $P^2 = \begin{bmatrix} 1 & 0 \\ -3 & 4 \end{bmatrix}$.

We can see row sums $= 1$ for $P^2_{2\times 2}$ also.

So, the statement is true for $m (> 0)$.

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Q.22. [TIFR'2013] If A and B are similar matrices then every eigenvector of A is an eigenvector of B.

Sol. FALSE

Similar matrices:- A and B are called similar matrices if

$$B = P^{-1}AP \text{ for some invertible matrix } P.$$

For an easy reasoning, suppose A is diagonalizable and B is the diagonalization of A. A and B are similar by the definition of diagonalizable: there exists an invertible matrix P such that $P^{-1}AP = B$, and B is a diagonal matrix.

Then the standard basis vectors are the eigenvectors of B, but clearly these need not be the eigenvectors of A.

Take $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $P = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, let

$$B = P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ but $B\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Note:- For similar matrices eigenvalues are same.

Q.23. [TIFR'2013] If a real square matrix A is similar to a diagonal matrix and satisfies $A^n = 0$ for some n, then A must be the zero matrix.

Sol. TRUE

$$\lambda^n = 0$$
$$\Rightarrow \lambda = 0$$
$$\Rightarrow \text{eigen values of } A \text{ are all zero.}$$

Since, A is similar to a diagonal matrix also, so A is a zero matrix.

Q.24. [TIFR'2013] If A is a complex $n \times n$ matrix with $A^2 = A$, then $\text{rank}(A) = \text{trace}(A)$.

Solution:- TRUE

$$A^2 - A = O_{n\times n}$$
$$\Rightarrow A(A - I) = O_{n\times n}$$
$$\Rightarrow A = O_{n\times n}, \quad A = I_{n\times n} = \begin{pmatrix} -i^2 & 0 \\ 0 & -i^2 \end{pmatrix}_{2\times n}, \text{ for } n=2.$$

Here $A^2 = A$. Here $\text{Rank}(A) = \text{Trace}(A) = 2$.

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ABSTRACT ALGEBRA

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Q.1. [TIFR'2010] Which of the following is false?

(a) Any abelian group of order 27 is cyclic.

(b) Any abelian group of order 14 is cyclic.

(c) Any abelian group of order 21 is cyclic.

(d) Any abelian group of order 30 is cyclic.

Solution:- (a)

$$14 = 2 \times 7$$
$$21 = 3 \times 7$$
$$30 = 2 \times 3 \times 5$$

But 27 can't be expressed like this.

Q.2. [TIFR'2010] A cyclic group of order 60 has

(a) 12 generators (b) 15 generators (c) 16 generators

(d) 20 generators.

Solution:- (c)

$$\phi(60) = 60\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$$
$$= 16.$$

Q.3. [TIFR'2011] T/F:- In the ring $\mathbb{Z}/8\mathbb{Z}$, the equation $x^2 \equiv 1$ has exactly 2 solutions.

Solution:- FALSE

$x = \{0,1,2,3,4,5,6,7\}$ given that $x^2 - 1 \equiv 0 \pmod 8$

Here $x = 1, 3, 5, 7$ satisfies the given congruence.

So, there are four solutions.

Alternative way:-

	x	0	1	2	3	4	5	6	7
$x^2 \equiv 0 \pmod 8$	x^2	0	1	4	1	0	1	4	1
$x^2 - 1 \equiv 0 \pmod 8$	$x^2 - 1$	-1	0	3	0	-1	0	3	0

$\Rightarrow$ The equation has exactly 4 solutions.

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Q.4. [TIFR '2011] T/F:- There exists a group with a proper subgroup isomorphic to itself.

Solution:- TRUE since $(\mathbb{Z},+)$ is a group.

$n\mathbb{Z}$ (for any $n \in \mathbb{N}$) be a subset of $\mathbb{Z}$.

Hence, $(n\mathbb{Z},+)$ be subgroup of $(\mathbb{Z},+)$.

$\Rightarrow$ But there may be corresponding element in $(n\mathbb{Z},+)$ to each element of $(\mathbb{Z},+)$.

$\Rightarrow$ They are isomorphic also.

Q.5. [TIFR'2011] The groups $\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ are

(a) isomorphic (b) abelian (c) non-abelian (d) Cyclic.

(b) Solution:- (Two groups are isomorphic if one can be considered a re-labeling of the other).

In $\mathbb{Z}_3 \times \mathbb{Z}_3$, every element except 0 has order 3, since $(a,b)+(a,b)+(a,b) = (a+a+a, b+b+b)$ and $1+1+1 = 2+2+2 = 0$ in $\mathbb{Z}_3$. In $\mathbb{Z}_9$ the element 1 has order 9. So, the algebric structures are different.

The groups $\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ are abelian.

Q.6. [TIFR'2011] T/F:- The symmetric group S_5 consisting of permutations on 5 symbols has an element of order 6.

Solution:- TRUE since S_5 is a symmetric group consisting of permutation of 5 symbols.

$$O(S_5) = 5! = 120.$$

it could be possible for any $a \in S_5$ s.t. $a^6 = e$.

$\Rightarrow$ any element of order 6 will be in S_5.

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[**Eisenstein's Criterion:-** Let p be a prime integer. Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x] \text{ and}$$

$a_n \not\equiv 0 \pmod{p}$, $a_i \equiv 0 \pmod{p}$ for $i = 0, 1, 2, \ldots, n-1$ with $a_0 \not\equiv 0 \pmod{p^2}$.
Then $f(x)$ is irreducible over $\mathbb{Q}$.

Example:- Test for irreducibility over $\mathbb{Q}$ for the following polynomials:
(i) $8x^3 + 6x^2 - 9x + 24$ (ii) $x^4 + 9x + 3$.

Solution:- (i) Yes Apply Eisenstein's Criterion with $p = 3$.
(ii) Yes Apply Eisenstein's Criterion with $p = 3$.]

Q.7. [TIFR'2010] Which of the following statement is false?
(a) The polynomial $x^2 + x + 1$ is irreducible in $\mathbb{Z}/2\mathbb{Z}[x]$.
(b) The polynomial $x^2 - 2$ is irreducible in $\mathbb{Q}[x]$.
(c) The polynomial $x^2 + 1$ is reducible in $\mathbb{Z}/5\mathbb{Z}[x]$.
(d) The polynomial $x^2 + 1$ is reducible in $\mathbb{Z}/7\mathbb{Z}[x]$.

Solution:- (d)

(a) For $x = 0, 1, 2$. $x^2 + x + 1 \equiv 0 \pmod{2}$ has no roots in $\mathbb{Z}/2\mathbb{Z}[x]$.

(b) By Eisenstein theorem, taking $p = 2$, the polynomial is irreducible over $\mathbb{Q}$.

(c) $x^2 + 1 \equiv 0 \pmod{5}$ has roots when $x = 2, 3$.
So, $x^2 + 1$ is irreducible in $\mathbb{Z}/5\mathbb{Z}[x]$.

(d) $x^2 + 1 \equiv 0 \pmod{7}$ has no real root.
So, $x^2 + 1$ is irreducible in $\mathbb{Z}/7\mathbb{Z}[x]$.

Q.8. [TIFR'2012] The number 2 is a prime in $\mathbb{Z}[i]$.

Solution:- $\mathbb{Z}[i] = a + bi$, $a, b \in \mathbb{Z}$

FALSE Now, $(1+i)(1-i) = 2$.
So, 2 is not prime in $\mathbb{Z}[i]$.

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Q.9. [TIFR'2012] T/F:- If H_1 and H_2 are subgroups of a group G then $H_1 \cdot H_2 = \{ h_1 h_2 \in G \mid h_1 \in H_1 , h_2 \in H_2 \}$ is a subgroup of G.

Sol. FALSE

Q.10. [TIFR' 2012] T/F:- Every subgroup of order 74 in a group of order 148 is normal.

Sol. TRUE

Apply this theorem:- If H is a subgroup of G with index 2, then H is normal.

Here, every subgroup of order 74 in a group of order 148 is normal.

Q.11. [TIFR'2012] T/F:- The automorphism group $\mathrm{Aut}(\mathbb{Z}/2 \times \mathbb{Z}/2)$ is abelian.

Sol. FALSE

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Q.12. [TIFR'2012]

Any automorphism of the group $\mathbb{Q}$ under addition is of the form $x \mapsto qx$ for some $q \in \mathbb{Q}$.

Sol. TRUE

Let $f: \mathbb{Q} \to \mathbb{Q}$ be such a homomorphism, and let $q = f(1)$, then, (i) For any $m \in \mathbb{Z}$, $f(m) = m f(1) = mq$

(ii) For any $r = m/n \in \mathbb{Q}$, you have

$$n f(r) = f(m) = mq \Rightarrow f(r) = q(m/n) = qr.$$

Q.13. [TIFR'2012] T/F:- The polynomial $X^8 + 1$ is irreducible in $\mathbb{R}[X]$

Sol. FALSE

For x^4+1, for $a, b, c, d \in \mathbb{R}$, $x^4+1 = (x^2+ax+b)(x^2+cx+d)$ then x^4+1 is reducible in $\mathbb{R}[X]$

$$x^8+1 = (x^4+1)^2 - 2x^4$$
$$= (x^4+1)^2 - (\sqrt{2}x^2)^2$$
$$= (x^4+1-\sqrt{2}x^2)(x^4+1-\sqrt{2}x^2)$$ is reducible in $\mathbb{R}[X]$.

Alt:- As x^4+1 is reducible, then $x^4+1 = f(x)g(x)$ where f and g are non-constant polynomials. So, $x^8+1 = f(x^2)g(x^2)$ is reducible too.

Q.14. [TIFR'2013] T/F:- Every countable group G has only countably many distinct subgroups.

Sol. FALSE

Let G be the direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$ indexed by $\mathbb{N}$. G is countably infinite, but each subset $A \subseteq \mathbb{N}$ generates a distinct subgroup $G_A = \{x \in G : x_n = 0 \ \forall \ n \in \mathbb{N} \setminus A\}$ of G, and $\mathbb{N}$ has uncountably many subsets.

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Q.15. [TIFR'2013] T/F:- All non-trivial proper subgroups of $(\mathbb{R}, +)$ are cyclic.

Solution:- FALSE

Counter

Example:- $(\mathbb{Q}, +)$ is not cyclic, which is indeed a non-trivial subgroup of $(\mathbb{R}, +)$.

Q.16. [TIFR'2013] Every infinite abelian group has at least one T/F:- element of infinite order.

Sol. FALSE

Here is an example of infinite abelian group such that all its elements are of finite order: —

Let $(\mathbb{Q}, +)$ denote the groups of rational numbers under addition, and consider it's subgroup $(\mathbb{Z}, +)$ of integers. Then any element from the group $\mathbb{Q}/\mathbb{Z}$ has elements of the form $\frac{p}{q} + \mathbb{Z}$ which is of order at-most q. Hence, it's of finite order.

Q.17. [TIFR'2013] T/F:- There is an element of order 51 in the multiplicative group $(\mathbb{Z}/103\mathbb{Z})$.

Sol. TRUE

Use Fermat's Little Theorem here.

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Q.18. [TIFR'2013] T/F:- Any normal subgroup of order 2 is contained in the center of the group.

Sol. TRUE Let K be a normal subgroup of order 2 in group G. We need to show K lies in the centre of G.

Since K is a normal subgroup of order 2, there is only one non-identity element, say $a \in K$. Then for all $g \in G$. We have that $gag^{-1} = e$ or $gag^{-1} = a$. If the latter occurs, then $K \subseteq Z(G)$ and we are done since $(gag^{-1})g = ga = ag$.

Assume that $gag^{-1} = e$. Then $ga = g$, then multiplying on the left by g^{-1}, we get $a = e$, a contradiction. Thus, $K \subseteq Z(G)$ as desired.

Q.19. [TIFR'2013] T/F:- The intervals $[0,1)$ and $(0,1)$ are homeomorphic.

Sol. FALSE

Note that, Compactness is a topological invariant, i.e., it is presented exactly by homeomorphisms.

Here the interval $(0,1)$ is not compact, so, the intervals $[0,1)$ and $(0,1)$ are not homeomorphic.

Remark:-

No two of the intervals of the spaces $(0,1)$, $(0,1]$ and $[0,1]$ are homeomorphic.

Q.20. [TIFR'2013] T/F:- Every homeomorphism of the 2-sphere to itself has a fixed point.

Sol. FALSE Counter Example:

The antipodal map $f: S^2 \to S^2$ defined by $f(x) = -x$ does not have a fixed point.

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TOPOLOGY

WORLD OF MATHEMATICS AND STATISTICS
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Q.1. [TIFR'2011] T/F:- If A and B are closed subsets of $[0,\infty)$, then $A+B = \{x+y \mid x \in A, y \in B\}$ is closed in $[0,\infty)$.

Solution:- TRUE

By the definition, $\bar{A} = A$

$\bar{B} = B$

$\Rightarrow \bar{A}+\bar{B} = A+B = \overline{(A+B)}$

$\Rightarrow A+B$ is closed in $[0,\infty)$.

Q.2. [TIFR'2011] T/F:- A bounded continuous function on $\mathbb{R}$ is uniformly continuous.

Sol. TRUE

Theorem:- If f is continuous on any closed and bounded interval then it is uniformly continuous.

Consequently, if f is continuous at every point of I then f is uniformly continuous on I.

Q.3. [TIFR'2012] T/F:- $A \subset \mathbb{R}$ and open then the interior of closure of $\overline{A^0}$ is A.

Sol. TRUE $A \subset \mathbb{R}$ and open then its interior is A^0

Now we know that closure of A set S is the smallest closed set containing S.

and $(\bar{\bar{S}}) = \bar{S}$.

Therefore, the closure of a set A^0 is the smallest closed set containing A. Then $(\overline{A^0}) = A$.

Q.4. [TIFR'2012] T/F:- If every continuous function on $X \subset \mathbb{R}^2$ is bounded, then X is compact.

Sol. TRUE

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Q.5. [TIFR'2012] T/F :- The graph of $xy = 1$ in $\mathbb{C}^2$ is connected.

Solution:- TRUE

Every value of x has a corresponding value of y for which $xy = 1$ except when $x = 0$, since $\mathbb{R}$ and $\mathbb{C}$ are fields. If Γ is the function that takes a point $x \in \mathbb{C}$ to the corresponding point $\left(x, \frac{1}{x}\right)$ on the graph $xy = 1$. Then Γ must have domain $\mathbb{C} - \{0\}$. But $\mathbb{C} - \{0\}$ is connected and $\Gamma : \mathbb{C} - \{0\} \to \mathbb{C}^2$ is continuous, so $\Gamma(\mathbb{C} - \{0\}) = \{(x, y) \in \mathbb{C}^2 \mid xy = 1\}$ is connected.

Q.6. [TIFR'2013] T/F:- There exists a complete metric on the open interval $(0,1)$ inducing the usual topology.

Sol. TRUE

Q.7. [TIFR'2013] T/F:- Let $f : X \to Y$ be a continuous map between metric spaces. If f is a bijection, then its inverse is also continuous.

Sol. FALSE

Example:- Lets choose the domain as $[0, 2\pi[$ in $\mathbb{R}$, formula is $f(\theta) = (\cos\theta, \sin\theta)$. range is in $\mathbb{R}^2$. This is a continuous map of that interval one to one onto a circle. But the converse is discontinuous.

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Q.8. [TIFR '2013] T/F:- If every differentiable function on a subset $X \subset \mathbb{R}^n$ (i.e., restriction of a differentiable function on a neighbourhood of X) is bounded, then X is compact.

Sol. TRUE

Q.9. [TIFR '2013] T/F:- Let X be a complete metric space such that distance between any two points is less than 1. Then X is compact.

Sol. FALSE

Take the closed unit ball of radius $\frac{1}{4}$ of any infinite dimensional Banach space. Then check that X is not compact.

AH coay:- $X = \mathbb{R}$, $d(x, y) = \frac{1}{2}$ if $x \neq y$

$d(x, y) = 0$ if $x = y$.

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SET, COMBINATORICS, PROBABILITY.

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Q.1. [TIFR'2010] The total number of subsets of a set of 6 elements are —

(a) 720 (b) 6^6 (c) 21 (d) None

Sol. (d)

The total number of subset of set of n elements $= 2^n$

Here, $n = 6$, so, ans. is $= 2^6 = 64$.

Q.2. [TIFR'2011] $\exists$ a set $A \subset \{1, 2, \ldots, 100\}$ with 65 elements such that 65 can't be expressed as a sum of two elements in A.

T/F:-

Sol. TRUE $o(A) = 65$.

Two elements x, y in A s.t. $x + y = 65$.
So, we can take all the element from 65 to 100 in A.
And if x is in A then $65 - x$ will not be in A, so we can select at least 30 elements from 1 to 64 in A to follow the condition.

Thus, we can have a set $A \subset \{1, 2, \ldots, 100\}$ with 65 elements such that $\nexists$ $x, y \in A$ such that $x + y = 65$.

Q.3. [TIFR'2011] Suppose a box contains three cards, one with both sides white, one with both sides black, and one with one side white and the other side black. If you pick a card at random, and the side facing you is white, then the probability that the other side is white is $1/2$. $\rightarrow$ T/F.

Sol. FALSE

Bayes theorem says:- For a sequence of mutually exclusive and exhaustive events $A_1, A_2, \ldots$ with $P(A_i) > 0 \ \forall \ i = 1, 2, 3, \ldots$ then

$$P(A_j | B) = \frac{P(A_j)\, P(B | A_j)}{\sum_{i=1}^{\infty} P(A_i)\, P(B | A_i)}$$

, where B is any other event with $P(B) > 0$.

$$\therefore P(C_1 | W) = \frac{P(C_1)\, P(W | C_1)}{P(C_1)\, P(W | C_1) + P(C_2) P(W | C_2) + P(C_3) P(W | C_3)}$$

$$= \frac{1/3 \cdot 1}{1/3 \cdot 1 + 1/3 \cdot 0 + 1/3 \cdot 1/2}$$

$$= 2/3.$$

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Q.4. [TIFR '2011] Let $m \le n$ be natural numbers. The number of injective maps from a set of cardinality m to a set of cardinality n is

(a) $m!$ (b) $n!$ (c) $(n-m)!$ (d) none.

Sol. (d) Let f be a function $\ni f: A \to B$, $O(A) = m$, $O(B) = n$.

So, the number of injections are $= {}^{n}P_{m}$ if $m \le n$

$$= \frac{n!}{(n-m)!}.$$

Q.5. [TIFR '2013] T/F:- The function $f: \mathbb{Z} \to \mathbb{R}$ defined by $f(n) = n^3 - 3n$ is injective.

Sol. FALSE

$f(a) = a^3 - 3a$, $f(b) = b^3 - 3b$.

Let $f(a) = f(b)$, considering.

$\Rightarrow a^3 - 3a = b^3 - 3b$

$\Rightarrow a^3 - b^3 = 3(a-b)$

$\Rightarrow (a-b)(a^2+ab+b^2) = 3(a-b)$

$\Rightarrow a^2+ab+b^2 = 3. \not\Rightarrow a = b$, So, $f(n)$ is not injective.

Q.6. [TIFR '2013] T/F:- There exists a continuous surjective map from the complex plane onto the non-zero reals.

Sol. FALSE

Such a continuous surjection does not exist.

Let $f: C \to \mathbb{R}\setminus\{0\}$ be continuous surjection.

Then $(f^{-1})(0,\infty)$ and $(f^{-1})(-\infty,0)$ are open, union of which is the whole of C.

$\Rightarrow C$ is disconnected.

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DIFFERENTIAL EQUATIONS

Q.1. [TIFR'2010]

The solution of the ordinary Differential Equation

$$\frac{dy}{dx} = y, \quad y(0) = 0$$

(i) is bounded (ii) is positive
(iii) is negative (iv) is zero

Solution:- (iv)

$$\int \frac{dy}{y} = \int dx$$

$$\Rightarrow \ln y = x$$

$$\Rightarrow y = e^{x+c}$$

$$\Rightarrow y = ke^{x}.$$

at $x=0$, $y=k$.

This ODE has zero or no solution.

Q.2. [TIFR'2010]

The number of solutions of the ODE

$$\frac{d^2y}{dx^2} - y = 0, \quad y(0) = 0, \quad y(\pi) = 1$$

(a) is 0 (b) is 1 (c) is 2 (d) none

Solution:- (b) $\frac{d^2y}{dx^2} + \lambda y = 0$ has the general solution

$$y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}, \text{ when } \lambda < 0.$$

Here, $\lambda = -1$, so $y = c_1 e^{x} + c_2 e^{-x}$

$$y(0) = 0 \Rightarrow c_1 + c_2 = 0$$

$$y(\pi) = 1 \Rightarrow c_1 e^{\pi} + c_2 e^{-\pi} = 1$$

Solving we get $c_1 = \frac{1}{e^{\pi} - e^{-\pi}}$, $c_2 = \frac{1}{e^{-\pi} - e^{\pi}}$

So, $y = \frac{e^{x}}{e^{\pi} - e^{-\pi}} - \frac{e^{-x}}{e^{\pi} - e^{-\pi}}$ is a unique solution of the given differential equation.

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Q.3. [TIFR '2011]

The differential equation

$$\frac{dy}{dx} = y^{1/3}, \quad y(0)=0 \text{ has}$$

(a) a unique solution
(b) no non-trivial solution
(c) finite number of solutions
(d) infinite number of solutions

Solution:- (d)

$$\int y^{-1/3} dy = \int dx$$

$$\Rightarrow \frac{3}{2} y^{2/3} = x + c$$

Since $y(0)=0$

So, c becomes undetermined.
i.e. 'c' can take any arbitrary values.
So, there will be infinite number of solutions.

Q.4. [TIFR '2011]

The space of solutions of infinitely differentiable functions satisfying the equation $y'' + y = 0$ is infinite dimensional.
$\longrightarrow$ T/F.

Solution:- False

$y''+y=0$ have the solution $y = c_1 \sin x + c_2 \cos x$.
The solution is infinitely differentiable but it is one dimensional i.e. it depends only on the values of x.

Q.5. [TIFR '2012]

T/F: Consider the differential equations (with y is a function of x)

(1) $\frac{dy}{dx} = y$, $y(0)=0$

(2) $\frac{dy}{dx} = |y|^{1/3}$, $y(0)=0$.

Then (1) has infinitely many solutions but (2) has finite number of solutions.

Solution:- False

(1) has zero solution.
(2) has infinite number of solutions.

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