SOLUTION TO

SHORT-ANSWER TYPE QUESTIONS

FROM

FIFTEENTH EDITION

Test of Mathematics at the 10 + 2 Level

Ctanujit Classes of Mathematics,
Statistics & Economics.
Kolkata, W. B.,8420253573
website : www.ctanujit.in

INDIAN STATISTICAL INSTITUTE

SOLUTION TO

Ctanujit Classes of Mathematics,
Statistics & Economics.
Kolkata, W. B.,8420253573
website : www.ctanujit.in

SHORT-ANSWER TYPE QUESTIONS

FROM

TOMATO SUBJECTIVE BOOK

TOTAL QUESTIONS - 189

TOTAL SOLUTIONS GIVEN - 149

* REST 40 PROBLEMS ARE LEFT AS YOUR EXERCISES *

1. A vessel contains x gallons of wine and another contains y gallons of water. From each vessel z gallons are taken out and transferred to the other. From the resulting mixture in each vessel, z gallons are again taken out and transferred to the other. If after the second transfer, the quantity of wine in each vessel remains the same as it was after the first transfer, then show that $z(x+y) = xy$.

2. Find the number of positive integers less than or equal to 6300 which are not divisible by 3, 5 and 7.

3. A troop 5 metres long starts marching. A soldier at the end of the file steps out and starts marching forward at a higher speed. On reaching the head of the column, he immediately turns around and marches back at the same speed. As soon as he reaches the end of the file, the troop stops marching, and it is found that the troop has moved by exactly 5 metres. What distance has the soldier travelled?

4. The following table gives the urban population in India and percentages of total population in rural and urban centres for the decades during 1901-81.

URBAN AND RURAL POPULATION OF INDIA: 1901-1981

Year	*Urban Population in million*	*Percentage of Total Population*	
		Rural	*Urban*
1901	25.8	89.0	11.0
1911	25.9	89.6	10.4
1921	28.0	88.7	11.3
1931	33.5	87.8	12.2
1941	44.1	85.9	14.1
1951	62.4	82.4	17.6
1961	78.9	81.7	18.3
1971	108.9	79.8	20.2
1981	162.2	76.3	23.7

Verify the following statements against the given data and classify each statement into one of the following categories: (A) True ; (B) False; (C) Does not necessarily follow from the given information.
[Note: Do not make any additional assumptions. Use only the information given.]

(i) The percent increase in urban population during 1901-81 is about 2.5 to 3 times the percent increase in total population in that period.

(ii) The density of population in urban centres has increased by 160% during 1951-81.

(iii) The largest rate of increase in urban population in a decade during 1901-1981 occurred in 1971-81.

(iv) The smallest rate of increase in urban population in a decade during 1931-1981 occurred in 1931-41.

(v) The relative degree of urbanization (i.e., change in the percentage of urban population) was highest in 1941-51 and 1971-81.

X 5. In a study of the disparities in the levels of living in rural areas among different states in India, estimates were obtained for the per-capita household consumption expenditure on all items per month, (denoted by PCE) in the years 1963-64 and 1973-74. They are presented in the table below.

(i) Which state has a PCE closest to the all-India PCE (a) in 1963-64? (b) in 1973-74?

(ii) Suppose the overall disparity among states is measured by the ratio of the largest to the smallest of the state PCEs in a year. Has disparity among states increased or decreased between 1963-64 and 1973-74?

(iii) By considering the ranks of the states according to PCE, separately for each year, find out which state has improved its rank most and which state has declined most between the two years.

PER CAPITA MONTHLY HOUSEHOLD CONSUMPTION EXPENDITURE IN RURAL AREAS BY STATES, 1963-64 AND 1973-74

State	PCE in 1963-64	PCE in 1973-74
Andhra Pradesh	20.91	50.69
Assam	26.28	52.01
Bihar	21.24	56.31
Gujarat	22.69	54.54
Himachal Pradesh	25.75	71.85
Jammu & Kashmir	27.99	54.14
Karnataka	20.35	52.29
Kerala	20.45	55.32
Madhya Pradesh	23.21	50.84
Maharashtra	21.75	52.91
Manipur	22.20	52.88
Orissa	19.47	42.61
Punjab & Haryana	29.11	75.09
Rajasthan	23.27	63.98
Tamil Nadu	23.52	47.68
Tripura	23.66	50.15
Uttar Pradesh	21.37	51.50
West Bengal	23.83	47.47
INDIA	22.38	55.90

X 6. The following table gives the distribution of the workers in a community by age, sex and type of work.

FREQUENCY DISTRIBUTION OF WORKERS BY AGE, SEX AND TYPE OF WORK

Age in Years (last birthday)	Type of Work: Manual, Male	Manual, Female	Non-manual, Male	Non-manual, Female
5-10	8	15	0	0
11-15	10	20	0	0
16-20	22	20	25	5
21-35	35	45	65	25
36-50	25	35	45	15
51-65	20	15	15	5
above 65	10	5	0	0

A similar table that was produced for the same community ten years ago is as follows:

FREQUENCY DISTRIBUTION OF WORKERS BY AGE AND TYPE OF WORK

Age in years (last birthday)	Type of Work: Manual	Non-manual
5-10	15	0
11-15	35	0
16-20	40	40
21-35	50	80
36-50	70	65
51-65	20	15
above 65	10	0

The break-up of workers by sex is not given in the old table. Assume that the male-female ratios in both the '35 and below' and 'above 35' age-groups have remained the same over the last ten years for manual as well as non-manual workers. In which of these two age-groups has the number of female manual workers increased more?

X 7. The following table gives the natural logarithm of the bodyweights (in Kg) for five individuals going through a weight-loss programme for five successive weeks.

Person \ Week	0	1	2	3	4	5
A	4.24	4.00	3.75	3.50	3.40	3.40
B	4.32	4.15	3.90	3.95	6.30	3.40
C	4.24	4.10	3.80	3.50	3.45	3.60
D	4.38	4.40	4.25	4.30	4.10	3.95
E	4.38	4.30	4.35	4.20	3.80	3.85

(i) For each week identify the individual for whom there are equal number of individuals with higher and lower bodyweights, respectively. Call this individual the *midperson* for that week.

(ii) Plot the logarithm of the bodyweight of the midperson for each week against the week (in plain paper).

(iii) What is your prediction for the logarithm of the bodyweight of the midperson for the sixth week? Give reasons.

(iv) Comment on any unusual observations that you find in the table.

(v) Write a brief report (in about five sentences) on how effective the weight-loss programme has been on different individuals.

8. In a club of 80 members, 10 members play none of the games Tennis, Badminton and Cricket. 30 members play exactly one of these three games and 30 members play exactly two of these games. 45 members play at least one of the games among Tennis and Badminton, whereas 18 members play both Tennis and Badminton. Determine the number of Cricket playing members.

9. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$, where $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are real numbers. We write $\boldsymbol{x} > \boldsymbol{y}$, if for some k, $1 \leq k \leq (n-1)$, $x_1 = y_1, x_2 = y_2, \ldots, x_k = y_k$, but $x_{k+1} > y_{k+1}$. Show that for $\boldsymbol{u} = (u_1, ..., u_n)$, $\boldsymbol{v} = (v_1, ..., v_n)$, $\boldsymbol{w} = (w_1, ..., w_n)$ and $\boldsymbol{z} = (z_1, ..., z_n)$, if $\boldsymbol{u} > \boldsymbol{v}$ and $\boldsymbol{w} > \boldsymbol{z}$, then $(\boldsymbol{u} + \boldsymbol{w}) > (\boldsymbol{v} + \boldsymbol{z})$.

10. We say that a sequence $\{a_n\}$ has property P, if there exists a positive integer m such that $a_n \leq 1$ for every $n \geq m$. For each of the following sequences, determine whether it has the property P or not. [Do not use any result on limits.]

(i)

$$a_n = \begin{cases} 0.9 + \frac{200}{n} & \text{if } n \text{ is even} \\ \frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$$

(ii)

$$a_n = \begin{cases} 1 + \frac{\cos \frac{n\pi}{2}}{n} & \text{if } n \text{ is even} \\ \frac{1}{n} & \text{if } n \text{ is odd.} \end{cases}$$

11. Let x and n be positive integers such that $1 + x + x^2 + \ldots + x^{n-1}$ is a prime number. Then show that n is a prime number.

12. Let

$$x_n = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n}.$$

Then show that

$$x_n \leq \frac{1}{\sqrt{3n+1}}, \quad \text{for all } n = 1, 2, 3, \ldots .$$

13. (i) In the identity

$$\frac{n!}{x(x+1)(x+2)\ldots(x+n)} = \sum_{k=0}^{n} \frac{A_k}{x+k},$$

prove that

$$A_k = (-1)^k \binom{n}{k}.$$

(ii) Deduce that:

$$\binom{n}{0}\frac{1}{1\cdot 2} - \binom{n}{1}\frac{1}{2\cdot 3} + \binom{n}{2}\frac{1}{3\cdot 4} - \ldots + (-1)^n \binom{n}{n}\frac{1}{(n+1)(n+2)} = \frac{1}{n+2}.$$

14. Show that

$$\frac{3}{1.2.4} + \frac{4}{2.3.5} + \frac{5}{3.4.6} + \ldots + \frac{n+2}{n(n+1)(n+3)}$$
$$= \frac{1}{6}\left[\frac{29}{6} - \frac{4}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right].$$

15. How many natural numbers less than 10^8 are there, whose sum of digits equals 7?

16. Suppose k, n are integers ≥ 1. Show that $(k \cdot n)!$ is divisible by $(k!)^n$.

17. If the coefficients of a quadratic equation $ax^2 + bx + c = 0, (a \neq 0)$, are all odd integers, show that the roots cannot be rational.

18. Let $D = a^2 + b^2 + c^2$, where a and b are successive positive integers and $c = ab$. Prove that $\sqrt{D}$ is an odd positive integer.

19. Prove that

$${}^nC_0 + {}^nC_3 + {}^nC_6 + \cdots + {}^nC_{3k} \leq \frac{1}{3}(2^n + 2),$$

where n is a positive integer and k is the largest integer for which $3k \leq n$.

20. Let $u_n = (3+\sqrt{5})^n + (3-\sqrt{5})^n$ for $n = 1, 2, \ldots$.

(i) Show that for each n, u_n is an integer.

(ii) Show that $u_{n+1} = 6u_n - 4u_{n-1}$ for all $n \geq 2$.

(iii) Use (ii) above, to show that u_n is divisible by 2^n.

21. For a natural number n, let $a_n = n^2 + 20$. If d_n denotes the greatest common divisor of a_n and a_{n+1}, then show that d_n divides 81.

22. Let $n \geq 2$ be an integer. Let m be the largest integer which is less than or equal to n, and which is a power of 2. Put $l_n =$ the least common multiple of $1, 2, \ldots, n$. Show that l_n/m is odd, and that for every integer $k \leq n, k \neq m, l_n/k$ is even. Hence prove that

$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

is not an integer.

23. By considering the expression $(1+\sqrt{2})^n + (1-\sqrt{2})^n$, where n is a positive integer, show that the integers $[(1+\sqrt{2})^n]$ are alternatively even and odd as n takes values $1, 2, \ldots$. Here for any real number x, $[x]$ denotes the greatest integer less than or equal to x.

24. If n is a positive integer greater than 1 such that $3n+1$ is perfect square, then show that $n+1$ is the sum of three perfect squares.

25. Show that for every positive integer n, $\sqrt{n}$ is either an integer or an irrational number.

26. Show that $2^{2n} - 3n - 1$ is divisible by 9 for all $n \geq 1$.

27. Suppose that the roots of $x^2 + px + q = 0$ are rational numbers and p, q are integers. Then show that the roots are integers.

28. Let $f(x)$ and $g(x)$ be two quadratic polynomials all of whose coefficients are rational numbers. Suppose $f(x)$ and $g(x)$ have a common irrational root. Show that $g(x) = rf(x)$ for some rational number r.

29. Show that for every positive integer n, 7 divides $3^{2n+1} + 2^{n+2}$.

30. Show that if n is *any* odd integer greater than 1, then $n^5 - n$ is divisible by 80.

31. If k is an odd positive integer, prove that for any integer $n \geq 1$, $1^k + 2^k + \cdots + n^k$ is divisible by $\frac{n(n+1)}{2}$.

32. Show that the number $11\ldots1$ with 3^n digits is divisible by 3^n.

33. Let k be a fixed odd positive integer. Find the minimum value of x^2+y^2, where x, y are nonnegative integers and $x+y=k$.

34. Let $f(x,y) = x^2+y^2$. Consider the region, including the boundary, enclosed by $y=\frac{x}{2}$, $y=-\frac{x}{2}$ and $x=y^2+1$. Find the maximum value of $f(x,y)$ in this region.

35. (a) Prove that, for any odd integer n, n^4 when divided by 16 always leaves remainder 1.

(b) Hence or otherwise show that we cannot find integers $n_1, n_2, \ldots, n_8$ such that

$$n_1^4+n_2^4+\ldots+n_8^4 = 1993.$$

36. Let $a_1, a_2, \cdots, a_n$ be n numbers such that each a_i is either 1 or -1. If

$$a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_na_1a_2a_3 = 0,$$

then prove that 4 divides n.

37. Suppose p is a prime number such that $(p-1)/4$ and $(p+1)/2$ are also primes. Show that $p=13$.

38. Show that if a prime number p is divided by 30, then the remainder is either a prime or is 1.

39. Two integers m and n are called *relatively prime* if the greatest common divisor of m and n is 1. Prove that among any five consecutive positive integers there is one integer which is relatively prime to the other four integers. (*Hint*: For any two positive integers $m<n$, any common divisor has to be less than or equal to $n-m$).

40. (i) If k and l are positive integers such that k divides l, show that for every positive integer m, $1+(k+m)l$ and $1+ml$ are relatively prime.

(ii) Consider the smallest number in each of the $\binom{n}{r}$ subsets (of size r) of $S=\{1,2,\ldots,n\}$. Show that the arithmetic mean of the numbers so obtained is $\frac{n+1}{r+1}$.

41. Find the number of rational numbers m/n, where m, n are relatively prime positive integers satisfying $m<n$ and $mn=25!$.

42. Let $f(x)$ be a polynomial with integer coefficients. Suppose that there exist distinct integers a_1, a_2, a_3, a_4, such that $f(a_1)=f(a_2)=f(a_3)=f(a_4)=3$. Show that there does not exist any integer b with $f(b)=14$.

43. Show that the equation

$$x^3 + 7x - 14(n^2 + 1) = 0$$

has no integral root for any integer n.

44. Show that if $n > 2$, then $(n!)^2 > n^n$.

45. Let $J = \{0, 1, 2, 3, 4\}$. For x, y in J define $x \oplus y$ to be the remainder of the usual sum of x and y after division by 5 and $x \odot y$ to be the remainder of the usual product of x and y after division by 5. For example, $4 \oplus 3 = 2$ while $4 \odot 2 = 3$. Find x and y in J, satisfying the following equations simultaneously:

$$(3 \odot x) \oplus (2 \odot y) = 2, \quad (2 \odot x) \oplus (4 \odot y) = 1.$$

46. A *function* f from a set A into a set B is a rule which assigns to each element x in A, a unique (one and only one) element (denoted by $f(x)$) in B. A function f from A into B is called an *onto* function, if for each element y in B there is some element x in A, such that $f(x) = y$. Now suppose that $A = \{1, 2, \ldots, n\}$ and $B = \{1, 2, 3\}$. Determine the total number of onto functions from A into B.

47. For a finite set A, let $|A|$ denote the number of elements in the set A.

(a) Let F be the set of all functions

$$f : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, k\} \quad (n \geq 3,\ k \geq 2)$$

satisfying

$$f(i) \neq f(i+1) \quad \text{for every } i,\ 1 \leq i \leq n-1.$$

Show that $|F| = k(k-1)^{n-1}$.

(b) Let $c(n, k)$ denote the number of functions in F satisfying $f(n) \neq f(1)$. For $n \geq 4$, show that

$$c(n, k) = k(k-1)^{n-1} - c(n-1, k).$$

(c) Using (b) prove that for $n \geq 3$,

$$c(n, k) = (k-1)^n + (-1)^n(k-1).$$

48. Find the number of ways in which 5 different gifts can be presented to 3 children so that each child receives at least one gift.

49. x red balls, y black balls and z white balls are to be arranged in a row. Suppose that any two balls of the same colour are indistinguishable. Given that $x + y + z = 30$, show that the number of possible arrangements is the largest for $x = y = z = 10$.

50. All the permutations of the letters a, b, c, d, e are written down and arranged in alphabetical order as in a dictionary. Thus the arrangement *abcde* is in the first position and *abced* is in the second position. What is the position of the arrangement *debac*?

51. (a) Given m identical symbols, say H's, show that the number of ways in which you can distribute them in k boxes marked $1, 2, \ldots\ldots, k$, so that no box goes empty is $\binom{m-1}{k-1}$.

(b) In an arrangement of m H's and n T's, an uninterrupted sequence of one kind of symbol is called a *run*. (For example, the arrangement $HHHTHH$ $TTTH$ of 6 H's and 4 T's opens with an H-run of length 3, followed successively by a T-run of length 1, an H-run of length 2, a T-run of length 3 and, finally, an H-run of length 1.)

Find the number of arrangements of m H's and n T's in which there are *exactly* k H-runs. [You may use (a) above.]

52. (i) Find the number of all possible ordered k-tuples of non-negative integers $(n_1, n_2, \ldots, n_k)$ such that $\sum_{i=1}^{k} n_i = 100$.

(ii) Show that the number of all possible ordered 4-tuples of non-negative integers (n_1, n_2, n_3, n_4) such that $\sum_{i=1}^{4} n_i \leq 100$ is $\binom{104}{4}$.

53. Show that the number of ways one can choose a set of distinct positive integers, each smaller than or equal to 50, such that their sum is odd, is 2^{49}.

54. Let $S = \{1, 2, \ldots, n\}$. Find the number of unordered pairs $\{A, B\}$ of subsets of S such that A and B are disjoint, where A or B or both may be empty.

55. A *partition* of a set S is formed by disjoint, nonempty subsets of S whose union is S. For example, $\{\{1, 3, 5\}, \{2\}, \{4, 6\}\}$ is a partition of the set $T = \{1, 2, 3, 4, 5, 6\}$ consisting of subsets $\{1, 3, 5\}$, $\{2\}$ and $\{4, 6\}$. However, $\{\{1, 2, 3, 5\}, \{3, 4, 6\}\}$ is not a partition of T.

If there are k nonempty subsets in a partition, then it is called a partition into k classes. Let S_k^n stand for the number of different partitions of a set with n elements into k classes.

(i) Find S_2^n.

(ii) Show that $S_k^{n+1} = S_{k-1}^n + kS_k^n$.

56. Show that the number of ways in which four distinct integers can be chosen from $1, 2, \ldots, n, (n \geq 7)$ such that no two are consecutive is equal to $\binom{n-3}{4}$.

57. How many 6-letter words can be formed using the letters A, B and C so that each letter appears at least once in the word?

58. In a certain game, 30 balls of k different colours are kept inside a sealed box. You are told only the value of k, but not the number of balls of each colour. Based on this, you have to guess whether it is possible to split the balls into 10 groups of 3 each, such that in each group the three balls are of different colours. Your answer is to be a simple YES or NO. You win or lose a point according as your guess is correct or not. For what values of k, you can say NO and be sure of winning? For what values of k, you can say YES and be sure of winning? Justify your solution.

59. Consider the set of points

$$S = \{(x, y) : x, y \text{ are non-negative integers } \leq n\}.$$

Find the number of squares that can be formed with vertices belonging to S and sides parallel to the axes.

60. Consider the set S of all integers between and including 1000 and 99999. Call two integers x and y in S to be in the same equivalence class if the digits appearing in x and y are the same. For example, if $x = 1010$, $y = 1000$ and $z = 1201$, then x and y are in the same equivalence class, but y and z are not. Find the number of distinct equivalence classes that can be formed out of S.

61. Solve

$$6x^2 - 25x + 12 + \frac{25}{x} + \frac{6}{x^2} = 0.$$

62. Consider the system of equations $x + y = 2$, $ax + y = b$. Find conditions on a and b under which

(i) the system has exactly one solution;

(ii) the system has no solution;

(iii) the system has more than one solution.

63. If any one pair among the straight lines

$$ax + by = a + b, \; bx - (a + b)y = -a, \; (a + b)x - ay = b$$

intersect, then show that the three straight lines are concurrent.

64. If $f(x)$ is a real-valued function of a real variable x, such that $2f(x) + 3f(-x) = 15 - 4x$ for all x, find the function $f(x)$.

65. Show that for all real x, the expression $ax^2 + bx + c$ (where a, b, c are real constants with $a > 0$), has the minimum value $\frac{(4ac-b^2)}{4a}$. Also find the value of x for which this minimum value is attained.

66. If c is a real number with $0 < c < 1$, then show that the values taken by the function $y = \frac{x^2+2x+c}{x^2+4x+3c}$, as x varies over real numbers, range over all real numbers.

67. Describe the set of all real numbers x which satisfy $2\log_{2x+3} x < 1$.

68. (i) Determine m so that the equation

$$x^4 - (3m+2)x^2 + m^2 = 0$$

has four real roots in arithmetic progression.

(ii) Let a and b be two real numbers. If the roots of the equation

$$x^2 - ax - b = 0$$

have absolute value less than one, show that each of the following conditions holds:
(i) $|b| < 1$, (ii) $a + b < 1$ and (iii) $b - a < 1$.

69. Suppose that the three equations $ax^2 - 2bx + c = 0$, $bx^2 - 2cx + a = 0$ and $cx^2 - 2ax + b = 0$ all have only positive roots. Show that $a = b = c$.

70. Suppose that all roots of the polynomial equation

$$x^4 - 4x^3 + ax^2 + bx + 1 = 0$$

are positive real numbers. Show that all the roots of the polynomial are equal.

71. Consider the following simultaneous equations in x and y:

$$\begin{aligned} x + y + axy &= a \\ x - 2y - xy^2 &= 0 \end{aligned}$$

where a is a real constant. Show that these equations admit real solutions in x and y.

72. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the equation whose roots are $\alpha - \frac{1}{\beta\gamma}$, $\beta - \frac{1}{\alpha\gamma}$ and $\gamma - \frac{1}{\alpha\beta}$,

73. Consider the equation $x^3 + Gx + H = 0$, where G and H are complex numbers. Suppose that this equation has a pair of complex conjugate roots. Show that both G and H are real.

74. The sum of squares of the digits of a three-digit positive number is 146, while the sum of the two digits in the unit's and the ten's place is 4 times the digit in the hundred's place. Further, when the number is written in the reverse order, it is increased by 297. Find the number.

75. Show that there is at least one real value of x for which $\sqrt[3]{x} + \sqrt{x} = 1$.

76. Find the set of all values of m such that $y = \frac{x^2-x}{1-mx}$ can take all real values.

77. For $x > 0$, show that $\frac{x^n-1}{x-1} \geq nx^{\frac{n-1}{2}}$, where n is a positive integer.

78. For real numbers x, y and z, show that

$$|x| + |y| + |z| \leq |x + y - z| + |y + z - x| + |z + x - y|.$$

79. Let $\theta_1, \theta_2, \ldots, \theta_{10}$ be any values in the closed interval $[0, \pi]$. Show that

$$F = (1 + \sin^2\theta_1)(1 + \cos^2\theta_1)(1 + \sin^2\theta_2)(1 + \cos^2\theta_2)\ldots\ldots$$

$$(1 + \sin^2\theta_{10})(1 + \cos^2\theta_{10}) \leq (\frac{9}{4})^{10}.$$

What is the maximum value attainable by F and at what values of $\theta_1, \theta_2, \ldots, \theta_{10}$, is the maximum value attained?

80. If a, b, c are positive numbers, then show that

$$\frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} + \frac{a^2+b^2}{a+b} \geq a + b + c.$$

81. Find all possible real numbers a, b, c, d, e which satisfy the following set of equations:

$$\begin{aligned} 3a &= (b+c+d)^3, \\ 3b &= (c+d+e)^3, \\ 3c &= (d+e+a)^3, \\ 3d &= (e+a+b)^3, \\ 3e &= (a+b+c)^3. \end{aligned}$$

82. Let a, b, c, d be positive real numbers such that $abcd = 1$. Show that

$$(1+a)(1+b)(1+c)(1+d) \geq 16.$$

83. If a and b are positive real numbers such that $a + b = 1$, prove that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}.$$

84. Show that there is exactly one value of x which satisfies the equation

$$2\cos^2(x^3 + x) = 2^x + 2^{-x}.$$

85. (i) Prove, from first principles, that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

for every positive integer n.

(ii) Prove that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

for every negative integer n.

86. Sketch the set $A \cap B$ in the Argand plane, where $A = \{z : |\frac{z+1}{z-1}| \leq 1\}$ and $B = \{z : |z| - \operatorname{Re} z \leq 1\}$.

87. Let $P(z) = az^2 + bz + c$, where a, b, c are complex numbers.

(a) If $P(z)$ is real for all real numbers z, show that a, b, c are real numbers.

(b) In addition to (a) above, assume that $P(z)$ is not real whenever z is not real. Show that $a = 0$.

88. A pair of complex numbers z_1, z_2 is said to have *property P* if for *every* complex number z, we can find *real numbers* r and s such that $z = rz_1 + sz_2$. Show that a pair z_1, z_2 has property P *if and only if* the points z_1, z_2 and 0 on the complex plane are *not* collinear.

89. Let a be a non-zero complex number such that $|a| \neq 1$. Let P be the point a in the complex plane, and let Q be the point $1/\overline{a}$. Let C_1 be the circle $\{z : |z| = 1\}$ and let C_2 be any circle passing through P and Q. Show that C_1 and C_2 intersect orthogonally. [*Two circles are said to* ***intersect orthogonally*** *if the tangents at a point of intersection are perpendicular to each other.*]

90. Draw the region of points (x, y) in the plane, which satisfy $|y| \leq |x| \leq 1$.

91. Show that a necessary and sufficient condition for the line $ax + by + c = 0$, where a, b, c are nonzero real numbers, to pass through the first quadrant is either $ac < 0$ or $bc < 0$.

92. Let a and b be real numbers such that the equations $2x+3y = 4$ and $ax-by = 7$ have exactly one solution. Then, show that the equations $12x - 8y = 9$ and $bx + ay = 0$ also have exactly one solution.

93. Let ABC be *any* triangle, right-angled at A, with D *any* point on the side AB. The line DE is drawn parallel to BC to meet the side AC at the point E. F is the foot of the perpendicular drawn from E to BC. If $AD = x_1, DB = x_2, BF = x_3, EF = x_4$ and $AE = x_5$, then show that

$$\frac{x_1}{x_5} + \frac{x_2}{x_5} = \frac{x_1x_3 + x_4x_5}{x_3x_5 - x_1x_4}.$$

94. Consider the circle of radius 1 with its centre at the point (0,1). From this initial position, the circle is rolled along the positive x-axis without slipping. Find the locus of the point P on the circumference of the circle which is on the origin at the initial position of the circle.

95. Let the circles

$$x^2 + y^2 - 2cy - a^2 = 0 \text{ and } x^2 + y^2 - 2bx + a^2 = 0,$$

with centres at A and B intersect at P and Q. Show that the points A, B, P, Q and $O = (0, 0)$ lie on a circle.

96. Two intersecting circles are said to be *orthogonal* to each other, if the tangents to the two circles at any point of intersection, are perpendicular to each other. Show that every circle through the points $(2, 0)$ and $(-2, 0)$ is orthogonal to the circle $x^2 + y^2 - 5x + 4 = 0$.

97. Consider the circle C whose equation is

$$(x - 2)^2 + (y - 8)^2 = 1$$

and the parabola P with the equation

$$y^2 = 4x.$$

Find the minimum value of the length of the segment AB as A moves on the circle C and B moves on the parabola P.

98. Let $f(x, y) = xy$, where $x \geq 0$ and $y \geq 0$. Prove that the function f satisfies the following property:

$$f(\lambda x + (1 - \lambda)x', \lambda y + (1 - \lambda)y') > \min\{f(x, y), f(x', y')\}$$

for all $(x, y) \neq (x', y')$ and for all $\lambda \in (0, 1)$.

99. If a circle intersects the hyperbola $y = 1/x$ at four distinct points $(x_i, y_i), i = 1, 2, 3, 4$, then prove that $x_1x_2 = y_3y_4$.

100. Consider the parabola $y^2 = 4x$. Let $P = (a, b)$ be any point inside the parabola, i.e., $b^2 < 4a$, and let F be the focus of the parabola. Find the point Q on the parabola such that $FQ + QP$ is minimum. Also, show that the normal to the parabola at Q bisects the angle FQP.

101. Let E be the ellipse with centre at origin O whose major and minor axes are of length $2a$ and $2b$ respectively. Let θ be the acute angle at which E is cut by a circle with centre at the origin (i.e., θ is the acute angle of intersection of their tangents at a point of intersection). Prove that the maximum possible value of θ is $tan^{-1}(\frac{a^2-b^2}{2ab})$.

102. Suppose that AB is an arc of a circle with a given radius and centre subtending an angle θ ($0 < \theta < \pi$ is fixed) at the centre. Consider an arbitrary point P on this arc and the product $l(AP) \cdot l(PB)$, where $l(AP)$ and $l(PB)$ denote the lengths of the straight lines AP and PB, respectively. Determine possible location(s) of P for which this product will be maximized. Justify your answer.

103. Let P be the fixed point (3,4) and Q the point $(x, \sqrt{25 - x^2})$. If $M(x)$ is the slope of the line PQ, find $\lim_{x \to 3} M(x)$.

104. Let A and B be two fixed points 3 cm apart.

(a) Let P be any point not collinear with A and B, such that $PA = 2PB$. The tangent at P to the circle passing through the points P, A and B meets the extended line AB at the point K. Find the lengths of the segments KB and KP.

(b) Hence or otherwise, prove that the locus of all points P in the plane such that $PA = 2PB$ is a circle.

105. Tangents are drawn to a given circle from a point on a given straight line, which does not meet the given circle. Prove that the locus of the mid-point of the chord joining the two points of contact of the tangents with the circle is a circle.

106. The circles C_1, C_2 and C_3 with radii 1, 2 and 3, respectively, touch each other externally. The centres of C_1 and C_2 lie on the x-axis, while C_3 touches them from the top. Find the ordinate of the centre of the circle that lies in the region enclosed by the circles C_1, C_2 and C_3 and touches all of them.

107. If a, b and c are the lengths of the sides of a triangle ABC and if p_1, p_2 and p_3 are the lengths of the perpendiculars drawn from the circumcentre onto the sides BC, CA and AB respectively, then show that

$$\frac{a}{p_1} + \frac{b}{p_2} + \frac{c}{p_3} = \frac{abc}{4p_1p_2p_3}.$$

108. Inside an equilateral triangle ABC, an arbitrary point P is taken from which the perpendiculars PD, PE and PF are dropped onto the sides BC, CA and AB, respectively. Show that the ratio $\dfrac{PD+PE+PF}{BD+CE+AF}$ does not depend upon the choice of the point P and find its value.

109. Let P be an interior point of the triangle ΔABC. Assume that AP, BP and CP meet the opposite sides BC, CA and AB ar D, E and F, respectively. Show that

$$\frac{AF}{FB}+\frac{AE}{EC}=\frac{AP}{PD}.$$

110. Let $ABCD$ be a cyclic quadrilateral with lengths of sides $AB=p$, $BC=q$, $CD=r$ and $DA=s$. Show that

$$\frac{AC}{BD}=\frac{ps+qr}{pq+rs}.$$

111. AB is a chord of a circle C.

(a) Find a point P on the circumference of C such that $PA.PB$ is the maximum.

(b) Find a point P on the circumference of C which maximises $PA+PB$.

112. A rectangle $OACB$ with the two axes as two sides, the origin O as a vertex is drawn in which the length OA is four times the width OB. A circle is drawn passing through the points B and C and touching OA at its mid-point, thus dividing the rectangle into three parts. Find the ratio of the areas of these three parts.

113. Find the vertices of the two right angled triangles each having area 18 and such that the point $(2,4)$ lies on the hypotenuse and the other two sides are formed by the x and y axes.

114. Let PQ be a line segment of a fixed length L with its two ends P and Q sliding along the X–axis and Y–axis respectively. Complete the rectangle $OPRQ$ where O is the origin. Show that the locus of the foot of the perpendicular drawn from R on PQ is given by

$$x^{2/3}+y^{2/3}=L^{2/3}.$$

115. If $\frac{\sin^4 x}{a}+\frac{\cos^4 x}{b}=\frac{1}{a+b}$, then show that $\frac{\sin^6 x}{a^2}+\frac{\cos^6 x}{b^2}=\frac{1}{(a+b)^2}$.

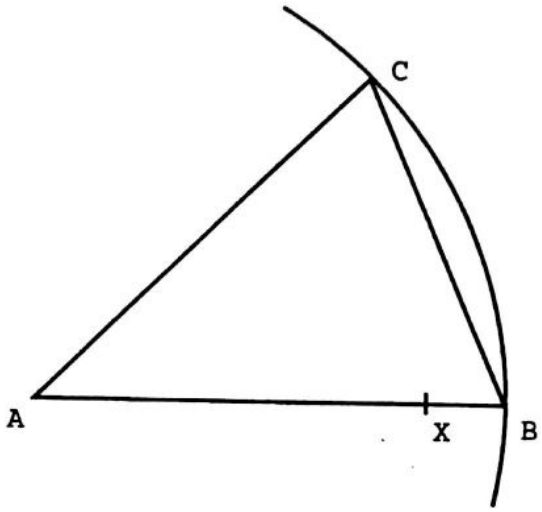
116. If A, B, C are the angles of a triangle, then show that $\sin A+\sin B-\cos C\leq\frac{3}{2}$.

117. Let $[x]$ denote the largest integer (positive, negative or zero) less than or equal to x. Let $y = f(x) = [x] + \sqrt{x - [x]}$ be defined for all real numbers x.

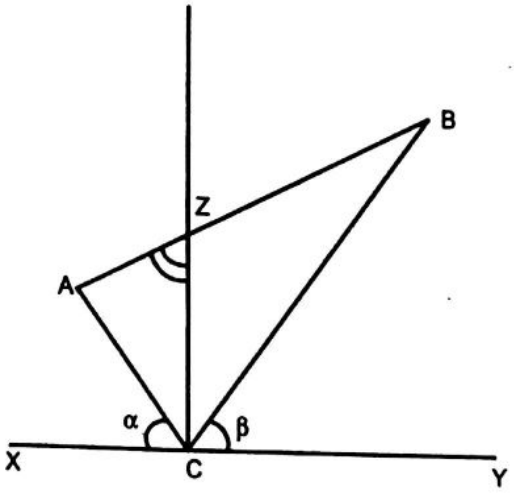
(i) Sketch on plain paper, the graph of the function $f(x)$ in the range $-5 \leq x \leq 5$.

(ii) Show that, given any real number y_0, there is a real number x_0, such that $y_0 = f(x_0)$.

118. Let X be a point on a straight line segment AB such that $AB \cdot BX = AX^2$. Let C be a point on the circle with centre at A and radius AB such that $BC = AX$. (See figure.) Show that the angle $BAC = 36°$.



119. In the adjoining figure CZ is perpendicular to XY and the ratio of the lengths AZ to ZB is 1:2. The angle ACX is α and the angle BCY is β. Find an expression for the angle AZC in terms of α and β.



120. (i) If $A + B + C = n\pi$ where n is a positive integer, show that

$$\sin 2A + \sin 2B + \sin 2C = (-1)^{n-1} 4 \sin A \sin B \sin C.$$

(ii) Let triangles ABC and DEF be inscribed in the same circle. If the triangles are of equal perimeter, then prove that

$$\sin A + \sin B + \sin C = \sin D + \sin E + \sin F.$$

(iii) State and prove the converse of (ii) above.

121. Let $\{x_n\}$ be a sequence such that $x_1 = 2, x_2 = 1$ and

$$2x_n - 3x_{n-1} + x_{n-2} = 0$$

for $n > 2$. Find an expression for x_n.

122. Sketch on plain paper, the graph of the function $y = \sin(x^2)$, in the range $0 \leq x \leq \sqrt{4\pi}$.

123. Let $[x]$ denote the largest integer less than or equal to x. For example, $[4\frac{1}{2}] = 4$; $[4] = 4$. Draw a rough sketch of the graphs of the following functions on plain paper:

(i) $f(x) = [x]$;
(ii) $g(x) = x - [x]$;
(iii) $h(x) = \frac{1}{[x]}$.

124. Sketch, on plain paper, the graph of $y = \frac{x^2+1}{x^2-1}$.

125. Let $f : \mathbb{N} \to \mathbb{N}$ be the function defined by $f(0) = 0, f(1) = 1$, and $f(n) = f(n-1) + f(n-2)$ for $n \geq 2$, where $\mathbb{N}$ is the set of all non-negative integers. Prove the following results:

(i) $f(n) < f(n+1)$ for all $n \geq 2$.
(ii) There exist precisely four non-negative integers n for which $f(f(n)) = f(n)$.
(iii) $f(5n)$ is divisible by 5, for all n.

126. Sketch, on plain paper, the regions represented on the plane by the following:

(i) $|y| = \sin x$;
(ii) $|x| - |y| \geq 1$.

127. Find all (x, y) such that

$$\sin x + \sin y = \sin(x + y) \quad \text{and} \quad |x| + |y| = 1.$$

128. Draw the graph (on plain paper) of

$$f(x) = \min\{|x| - 1, |x - 1| - 1, |x - 2| - 1\}.$$

129. Using calculus, sketch the graph of the following function on a plain paper:

$$f(x) = \frac{5 - 3x^2}{1 - x^2}.$$

130. (a) Study the derivatives of the function

$$f(x) = \frac{x+1}{(x-1)(x-7)}$$

to make conclusions about the behaviour of the function as x ranges over all possible values for which the above formula for $f(x)$ is meaningful.

(b) Use the information obtained in (a) to draw a rough sketch of the graph of $f(x)$ on plain paper.

131. Sketch the curve $y = 4x^3 - 3x + a$ on plain paper and show that the equation (in x)

$$4x^3 - 3x + a = 0,$$

(where the real constant a is such that $0 < |a| < 1$) has three distinct real roots all of which have their absolute values smaller than 1.

132. For the following function f study its derivatives and use them to sketch its graph on plain paper.

$$f(x) = \frac{x-1}{x+1} + \frac{x+1}{x-1} \quad \text{for} \quad x \neq -1, 1.$$

133. Use the derivatives and the left and right limits at points of discontinuities, if any, of the function

$$f(x) = \frac{x}{x+2} + \frac{x+2}{x}$$

to make conclusions about the behaviour of the function as x ranges over all possible values. Using this, draw a rough sketch of the graph of the function $f(x)$ on plain paper.

134. Using Calculus, sketch on plain paper the graph of the function

$$f(x) = x^2 + x + \frac{1}{x} + \frac{1}{x^2} \quad \text{for} \quad x \neq 0.$$

Show that the function f defined as above for *positive real* numbers attains a unique minimum. What is the minimum value of the function? What is the value of x at which the minimum is attained?

135. Suppose $f(x)$ is a continuous function such that $f(x) = \int_0^x f(t)dt$. Prove that $f(x)$ is identically equal to zero.

136. Consider the function

$$f(x) = \begin{cases} \sin\left(\frac{\pi}{4}(x - [x])\right) & \text{if } [x] \text{ is odd, } x \geq 0 \\ \cos\left(\frac{\pi}{4}(1 - x + [x])\right) & \text{if } [x] \text{ is even, } x \geq 0 \end{cases}$$

where $[x]$ denotes the largest integer smaller than or equal to x.

(i) Sketch the graph of the function f on plain paper.

(ii) Determine the points of discontinuities of f and the points where f is not differentiable.

137. For a real number x, let $[x]$ denote the largest integer less than or equal to x and $< x >$ denote $x - [x]$. Find all the solutions of the equation

$$13[x] + 25 < x >= 271.$$

138. For any positive integer n, let $\langle n \rangle$ denote the integer nearest to $\sqrt{n}$.

(a) Given a positive integer k, describe all positive integers n such that $\langle n \rangle = k$.

(b) Show that

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n} = 3.$$

139. Find all *positive* integers x such that $[x/5] - [x/7] = 1$, where, for any real number t, $[t]$ is the greatest integer less than or equal to t.

140. Consider the function

$$f(x) = \lim_{n \to \infty} \frac{\log_e(2 + x) - x^{2n} \sin x}{1 + x^{2n}}, \quad x > 0.$$

(i) Is $f(x)$ continuous at $x = 1$? Justify your answer.

(ii) Show that $f(x)$ does not vanish anywhere in the interval $0 \leq x \leq \frac{\pi}{2}$, and indicate the points where $f(x)$ changes its sign.

141. Consider the function $f(t) = e^{-\frac{1}{t}}, t > 0$. Let for each positive integer n, P_n be the polynomial such that $\frac{d^n}{dt^n} f(t) = P_n(\frac{1}{t})e^{-\frac{1}{t}}$ for all $t > 0$. Show that

$$P_{n+1}(x) = x^2(P_n(x) - \frac{d}{dx}P_n(x)).$$

142. Study the derivative of the function

$$f(x) = x^3 - 3x^2 + 4,$$

and roughly sketch the graph of $f(x)$, on plain paper.

143. Study the derivative of the function

$$f(x) = \log_e x - (x - 1), \text{for } x > 0,$$

and roughly sketch the graph of $f(x)$, on plain paper.

144. Suppose f is a real-valued differentiable function defined on $[1, \infty)$ with $f(1) = 1$. Suppose, moreover, that f satisfies

$$f'(x) = \frac{1}{x^2 + f^2(x)}.$$

Show that $f(x) \leq 1 + \pi/4$ for every $x \geq 1$.

145. Let $f(x)$ be a real valued function of a variable x such that $f'(x)$ takes both positive and negative values and $f''(x) > 0$ for all x. Show that there is a real number p such that $f(x)$ is an increasing function of x for all $x \geq p$.

146. Suppose f is a function such that $f(x) > 0$ and $f'(x)$ is continuous at every real number x. If $f'(t) \geq \sqrt{f(t)}$ for all t, then show that

$$\sqrt{f(x)} \geq \sqrt{f(1)} + \frac{1}{2}(x - 1).$$

for all $x \geq 1$.

147. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called *periodic* if for some constant $a > 0$, $f(x + a) = f(x)$ for every real number x. Show that the function

$$f(x) = \cos\ x + \cos(\frac{\sqrt{3}}{2}x)$$

is not periodic.

148. Show that there is no real constant $c > 0$ such that $\cos\sqrt{x+c} = \cos\sqrt{x}$ for all real numbers $x \geq 0$.

149. Consider the real-valued function f, defined over $(-\infty, \infty)$ by $f(x) = x^4 + bx^3 + cx^2 + dx + e$, where b, c, d, e are real numbers and $3b^2 < 8c$. Show that the function f has a unique minimum.

150. Find the maximum among $1, 2^{1/2}, 3^{1/3}, 4^{1/4}, \cdots$.

151. Let x be a positive number. A sequence $\{x_n\}$ of real numbers is defined as follows:

$$x_1 = \frac{1}{2}(x + \frac{5}{x}), \quad x_2 = \frac{1}{2}(x_1 + \frac{5}{x_1}), \ldots, \text{and in general,}$$

$$x_{n+1} = \frac{1}{2}(x_n + \frac{5}{x_n}) \quad \text{for all } n \geq 1.$$

(a) Show that, for all $n \geq 1$, $\frac{x_n - \sqrt{5}}{x_n + \sqrt{5}} = (\frac{x - \sqrt{5}}{x + \sqrt{5}})^{2^n}$.

(b) Hence find $\lim_{n\to\infty} x_n$.

152. Let a_0 and b_0 be any two positive integers. Define a_n, b_n for $n \geq 1$ using the relations $a_n = a_{n-1} + 2b_{n-1}$, $b_n = a_{n-1} + b_{n-1}$ and let $c_n = \frac{a_n}{b_n}$, for $n = 0, 1, 2, \ldots$.

(a) Write $(\sqrt{2} - c_{n+1})$ in terms of $(\sqrt{2} - c_n)$.

(b) Show that $|\sqrt{2} - c_{n+1}| < \frac{1}{1+\sqrt{2}}|\sqrt{2} - c_n|$.

(c) Show that $\lim_{n\to\infty} c_n = \sqrt{2}$.

153. Suppose $x_1 = \tan^{-1} 2 > x_2 > x_3 > \cdots$ are positive real numbers satisfying

$$\sin(x_{n+1} - x_n) + 2^{-(n+1)} \sin x_n \sin x_{n+1} = 0 \qquad \text{for } n \geq 1.$$

Find $\cot x_n$. Also, show that $\lim_{n\to\infty} x_n = \frac{\pi}{4}$.

154. Find the maximum and minimum values of the function $f(x) = x^2 - x\sin x$, in the closed interval $[0, \frac{\pi}{2}]$.

155. Evaluate

$$\lim_{n\to\infty} \{\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}\}.$$

156. A man walking towards a building, on which a flagstaff is fixed vertically, observes the angle subtended by the flagstaff to be the greatest when he is at a distance d from the building. If θ is the observed greatest angle, show that the length of the flagstaff is $2d\tan\theta$.

157. Evaluate

$$\lim_{n\to\infty}\{(1+\frac{1}{2n})(1+\frac{3}{2n})(1+\frac{5}{2n})\dots(1+\frac{2n-1}{2n})\}^{\frac{1}{2n}}.$$

158. Find the value of

$$\int_2^{11}\frac{dx}{1-x}.$$

159. If $A=\int_0^{\pi}\frac{\cos x}{(x+2)^2}dx$, then show that

$$\int_0^{\pi/2}\frac{\sin x\cos x}{(x+1)}dx=\frac{1}{2}(\frac{1}{2}+\frac{1}{\pi+2}-A).$$

160. Prove by induction or otherwise that

$$\int_0^{\pi/2}\frac{\sin(2n+1)x}{\sin x}dx=\frac{\pi}{2}$$

for every integer $n\geq 0$.

161. Show that

$$\int_0^{\pi}\left|\frac{\sin nx}{x}\right|dx\geq\frac{2}{\pi}\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right).$$

162. Show that

$$2(\sqrt{251}-1)<\sum_{k=1}^{250}\frac{1}{\sqrt{k}}<2(\sqrt{250}).$$

163. Using the identity $\log x=\int_1^x\frac{dt}{t}$, $x>0$, or otherwise, prove that

$$\frac{1}{n+1}\leq\log(1+\frac{1}{n})\leq\frac{1}{n}$$

for all integers $n\geq 1$.

164. Show that the area of the bounded region enclosed between the curves $y^3 = x^2$ and $y = 2 - x^2$, is $2\frac{2}{15}$.

165. Find the area of the region in the xy-plane, bounded by the graphs of $y = x^2, x + y = 2$ and $y = -\sqrt{x}$.

166. A cow is grazing with a rope around her neck and the other end of the rope is tied to a pole. The length of the rope is 10 metres. There are two boundary walls perpendicular to each other, one at a distance of 5 metres to the east of the pole and another at a distance of $5\sqrt{2}$ metres to the north of the pole. Find the area the cow can graze on.

167. Show that the larger of the two areas into which the circle $x^2 + y^2 = 64$ is divided by the curve $y^2 = 12x$ is $\frac{16}{3}(8\pi - \sqrt{3})$.

168. Out of a circular sheet of paper of radius a, a sector with central angle θ is cut out and folded into the shape of a conical funnel. Show that the volume of the funnel is maximum when θ equals $2\pi\sqrt{\frac{2}{3}}$.

169. A *regular* five-pointed star is inscribed in a circle of radius r. (See the figure.) Show that the area of the region inside the star is $\frac{10r^2 \tan(\pi/10)}{3-\tan^2(\pi/10)}$.

170. Let $\{C_n\}$ be an infinite sequence of circles lying in the positive quadrant of the XY–plane, with strictly decreasing radii and satisfying the following conditions. Each C_n touches both the X–axis and the Y–axis. Further, for all $n \geq 1$, the circle C_{n+1} touches the circle C_n externally. If C_1 has radius 10 cm, then show that the sum of the areas of all these circles is $\frac{25\pi}{3\sqrt{2}-4}$ sq cm.

171. Let ABC be an isosceles triangle with $AB = BC = 1$ cm and $\angle A = 30°$. Find the volume of the solid obtained by revolving the triangle about the line AB.

172. Suppose there are k teams playing a round robin tournament; that is, each team plays against all the other teams and no game ends in a draw. Suppose the ith team loses l_i games and wins w_i games. Show that

$$\sum_{i=1}^{k} l_i^2 = \sum_{i=1}^{k} w_i^2.$$

173. Let $P_1, P_2, \ldots, P_n$ be polynomials in x, each having all integer coefficients, such that $P_1 = P_1^2 + P_2^2 + \ldots + P_n^2$. Assume that P_1 is not the zero polynomial. Show that $P_1 = 1$ and $P_2 = P_3 = \ldots = P_n = 0$.

174. Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c and d are integers. The sums of the pairs of roots of $P(x)$ are given by 1, 2, 5, 6, 9 and 10. Find $P(\frac{1}{2})$.

175. Let $P(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots\cdots + a_1x + a_0$ be a polynomial with integer coefficients, such that, $P(0)$ and $P(1)$ are odd integers. Show that:

(a) $P(x)$ does not have any even integer roots.

(b) $P(x)$ does not have any odd integer roots.

176. Suppose that $P(x)$ is a polynomial of degree n such that

$$P(k) = \frac{k}{k+1} \quad \text{for} \quad k = 0, 1, \ldots, n.$$

Find the value of $P(n+1)$.

177. There are 1000 doors $D_1, D_2, \ldots, D_{1000}$ and 1000 persons $P_1, P_2, \ldots, P_{1000}$. Initially all the doors were closed. Person P_1 goes and opens all the doors. Then person P_2 closes doors $D_2, D_4, \ldots, D_{1000}$ and leaves the odd-numbered doors open. Next, P_3 changes the state of every third door, that is, $D_3, D_6, \ldots, D_{999}$. (For instance, P_3 closes the open door D_3 and opens the closed door D_6, and so on.) Similarly, P_m changes the state of the doors $D_m, D_{2m}, D_{3m}, \ldots, D_{nm}, \ldots$ while leaving the other doors untouched. Finally, P_{1000} opens D_{1000} if it were closed and closes it if it were open. At the end, how many doors remain open?

178. Let l, b be positive integers. Divide the $l \times b$ rectangle into lb unit squares in the usual manner. Consider one of the two diagonals of this rectangle. How many of these unit squares contain a segment of positive length of this diagonal?

179. Let $X = \{0, 1, 2, 3, \ldots, 99\}$. For a, b in X, we define $a * b$ to be the remainder obtained by dividing the product ab by 100. For example, $9 * 18 = 62$ and $7 * 5 = 35$. Let x be an element in X. An element y in X is called the inverse of x if $x * y = 1$. Find which of the elements $1, 2, 3, 4, 5, 6, 7$ have inverses and write down their inverses.

180. Each pair in a group of 20 persons is classified by the existence of kinship relation and friendship relation between them. The following table of data is obtained from such a classification

KINSHIP AND FRIENDSHIP RELATION AMONG 20 PERSONS

Friendship → *Kinship* ↓	Yes	No
Yes	27	31
No	3	129

Determine (with justifications) whether each of the following statements is supported by the above data:

(i) Most of the friends are kin.

(ii) Most of the kin are friends.

181. Suppose that one moves along the points (m, n) in the plane where m and n are integers in such a way that each move is a diagonal step, that is, consists of one unit to the right or left followed by one unit either up or down.

(a) Which points (p, q) can be reached from the origin?

(b) What is the minimum number of moves needed to reach such a point (p, q)?

182. In a competition, six teams A, B, C, D, E, F play each other in the preliminary round—called *round robin* tournament. Each game ends either in a win or a loss. The winner is awarded two points while the loser is awarded zero points. After the round robin tournament, the three teams with the highest scores move to the final round. Based on the following information, find the score of each team at the end of the round robin tournament.

(i) In the game between E and F, team E won.

(ii) After each team had played four games, team A had 6 points, team B had 8 points and team C had 4 points. The remaining matches yet to be played were

(i) between A and D;

(ii) between B and E; and

(iii) between C and F.

(iii) The teams D, E and F had won their games against A, B and C respectively.

(iv) Teams A, B and D had moved to the final round of the tournament.

183. Let $N = \{1, 2, ..., n\}$ be a set of elements called voters. Let $\mathcal{C} = \{S : S \subseteq N\}$ be the set of all subsets of N. Members of $\mathcal{C}$ are called coalitions. Let f be a function from $\mathcal{C}$ to $\{0, 1\}$. A coalition $S \subseteq N$ is said to be *winning* if $f(S) = 1$; it is said to be a *losing* coalition if $f(S) = 0$. Such a function f is called a voting game if the following conditions hold.

(a) N is a winning coalition.
(b) The empty set ϕ is a losing coalition.
(c) If S is a winning coalition and $S \subseteq S'$, then S' also is winning.
(d) If both S and S' are winning coalitions, then $S \bigcap S' \neq \phi$, i.e., S and S' have a common voter.

Show that the maximum number of winning coalitions of a voting game is 2^{n-1}. Also, find a voting game for which the number of winning coalitions is 2^{n-1}.

184. Let S be the set of all sequences $(a_1, a_2, \ldots)$ of non-negative integers such that
(i) $a_1 \geq a_2 \geq \ldots$; and
(ii) there exists a positive integer N such that $a_n = 0$ for all $n \geq N$.
Define the dual of the sequence $(a_1, a_2, \ldots)$ belonging to S to be the sequence $(b_1, b_2, \ldots)$, where, for $m \geq 1$, b_m is the number of a_n's which are greater than or equal to m.

(i) Show that the dual of a sequence in S belongs to S.

(ii) Show that the dual of the dual of a sequence in S is the original sequence itself.

(iii) Show that the duals of distinct sequences in S are distinct.

185. An operation $*$ on a set G is a mapping that associates with every pair of elements a and b of the set G, a unique element $a * b$ of G. G is said to be a *group under the operation* $*$, if the following conditions hold:
(i) $(a * b) * c = a * (b * c)$ for all elements a, b and c of G;
(ii) there is an element e of G such that $a * e = e * a = a$ for all elements a of G; and
(iii) for each element a of G, there is an element a' of G such that $a * a' = a' * a = e$.

If G is the set whose elements are all subsets of a set X, and, if $*$ is the operation on G defined as $A * B = (A \cup B) \setminus (A \cap B)$, show that G is a group under $*$.

(For any two subsets C and D of X, $C \setminus D$ denotes the set of all those elements which are in C but not in D).

186. At time 0, a particle is at the point 0 on the real line. At time 1, the particle divides into two and instantaneously after division, one particle moves 1 unit to the left and the other moves one unit to the right. At time 2, each of these particles divides into two, and one of the two new particles moves one unit to the left and the other moves one unit to the right. Whenever two particles meet, they destroy each other leaving nothing behind. How many particles will be there after time $2^{11} + 1$?

187. Suppose S={0,1} with the following addition and multiplication rules:

$$0+0=1+1=0 \qquad 0\cdot 0=0\cdot 1=1\cdot 0=0$$
$$0+1=1+0=1 \qquad 1\cdot 1=1$$

A system of polynomials is defined with coefficients in S. The sum and product of two polynomials in the system are the usual sum and product, respectively, where for the addition and multiplication of coefficients the above mentioned rules apply. For example, in the system,

$$(x+1)\cdot(x^2+x+1)=x^3+(1+1)x^2+(1+1)x+1=x^3+0x^2+0x+1=x^3+1.$$

Show that in this system x^3+x+1 is not factorizable, that is, one cannot write

$$x^3+x+1=(ax+b)\cdot(cx^2+dx+e),$$

where a, b, c, d and e are elements of S.

188. Consider the squares of an 8×8 chessboard filled with the numbers 1 to 64 as in the figure below. If we choose 8 squares with the property that there is exactly one from each row and exactly one from each column, and add up the numbers in the chosen squares, show that the sum obtained is always 260.

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

189. Let $a_1, a_2, \ldots, a_{100}$ be real numbers, each less than one, satisfy

$$a_1+a_2+\cdots+a_{100}>1.$$

(i) Let n_0 be the smallest integer n such that

$$a_1 + a_2 + \cdots + a_n > 1.$$

Show that all the sums a_{n_0}, $a_{n_0} + a_{n_0-1}$, $\ldots$, $a_{n_0} + \cdots + a_1$ are positive.

(ii) Show that there exist two integers p and q, $p < q$, such that the numbers

$$a_q, a_q + a_{q-1}, \ldots, a_q + \cdots + a_p,$$
$$a_p, a_p + a_{p+1}, \ldots, a_p + \cdots + a_q$$

are all positive.

SHORT ANSWER TYPE QUESTIONS

Q.1. Hints:-

$$\frac{x-z}{x} = \frac{z}{y}$$

$$\Rightarrow xy - zy = zx$$

$$\Rightarrow xy = z(x+y) \quad [\text{Proved}]$$

Q.2. Solution:- $S = \{1, 2, 3, \ldots\ldots, 6300\}$

Let A : Set of integers divisible by 3
B : " " " " " 5
C : " " " " " 7

We are to find :- $n(S) - n(A \cup B \cup C)$

$$= n(S) - \left[n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)\right]$$

$$= 6300 - \left\{\left[\frac{6300}{3}\right] + \left[\frac{6300}{5}\right] + \left[\frac{6300}{7}\right] - \left[\frac{6300}{3\times5}\right] - \left[\frac{6300}{5\times7}\right] - \left[\frac{6300}{3\times7}\right] + \left[\frac{6300}{3\times5\times7}\right]\right\}$$

i.e. $n(A \cup B \cup C)^c = 2880$. (Answer)

Q.3. Solution:- Let Vel. of troop $= v_t$
Vel. of soldier $= v_s$

Now, distance travelled by troop, when the soldier reached that point $= x$ m.

$\therefore$ Time taken $= \dfrac{x}{v_t} = \dfrac{\text{distance}}{\text{Velocity}}$.

So, distance travelled by soldier $=$ length of troop $+ x$
$= x + 5$.

$\therefore$ Time taken $= \dfrac{x+5}{v_s}$.

Given that $\dfrac{x}{v_t} = \dfrac{x+5}{v_s} \Rightarrow \dfrac{v_s}{v_t} = \dfrac{x+5}{x}$ ..... (i)

Again, the soldier retreated x units.

$\therefore$ time taken by him to retreat $= \frac{x}{v_s}$

the troop travelled $= \{10 - (5+x)\}$ m $= (5-x)$ m.

$\therefore$ time taken by them to travel $= \frac{5-x}{v_t}$.

So, by the question, we have $\frac{x}{v_s} = \frac{5-x}{v_t}$

$$\Rightarrow \frac{v_s}{v_t} = \frac{x}{5-x} \quad \text{.....(ii)}$$

Equating (i) and (ii), we have $\frac{x+5}{x} = \frac{x}{5-x}$

$$\Rightarrow x = \frac{5}{\sqrt{2}}.$$

$\therefore$ The soldier travelled $= 5 + x + x$

$= (5 + 5\sqrt{2})$ m. (Ans)

Q.18. Solution:- Let $a = 2n$

$b = 2n+1$

$c = 2n(2n+1)$

$= 4n^2 + 2n$

$D = a^2 + b^2 + c^2$

$\therefore D = 16n^4 + 16n^3 + 12n^2 + 4n + 1$

$D - 1 = 2(8n^4 + 8n^3 + 6n^2 + 2n)$

$\therefore D-1$ is an even number,

$\therefore D$ is an odd number,

So, $\sqrt{D}$ is an odd number,

Also, $D = 16n^4 + 16n^3 + 12n^2 + 4n + 1$

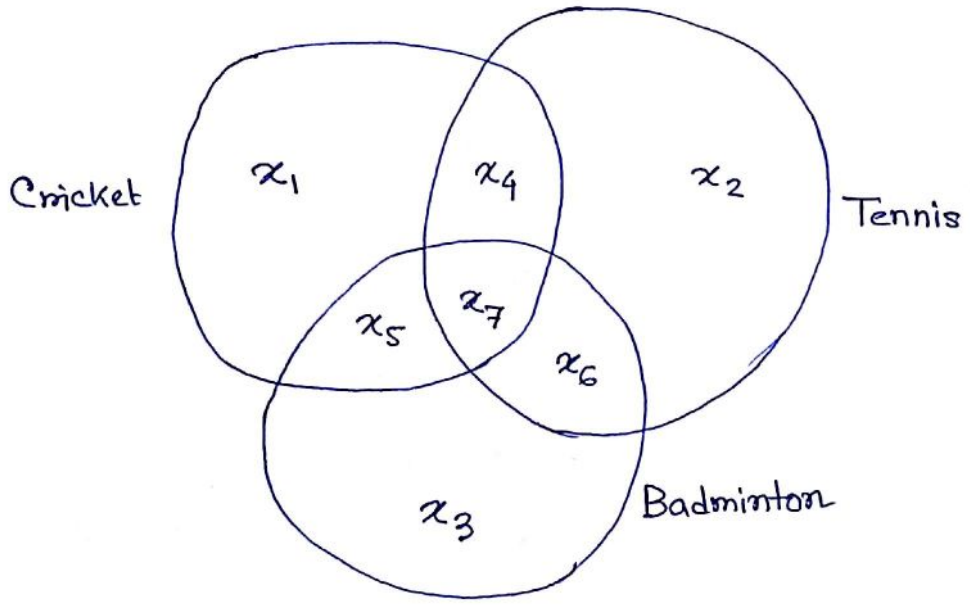
$= (4n^2 + 2n + 1)^2$

$\therefore \sqrt{D} = 4n^2 + 2n + 1$

$= 2(2n^2 + n) + 1$

$\therefore \sqrt{D}$ is an odd positive number.

Q.8. Solution:-



We get, $x_1 + x_2 + x_3 = 30$ and $x_4 + x_5 + x_6 = 30$.

Out of 80 members 10 members play none of the games.

$\Rightarrow$ $80 - 10 = 70$ members play games.

$\therefore$ $x_7 = 70 - (x_1 + x_2 + x_3) - (x_4 + x_5 + x_6) = 10$.

Now, 45 members play at least one of the games among Tennis and Badminton.

$\Rightarrow$ $70 - x_1 = 45$.

$\Rightarrow$ $x_1 = 25$.

Now, 18 members play both Tennis and Badminton.

$\Rightarrow$ $x_6 + x_7 = 18$

$\Rightarrow$ $x_6 = 8$.

$\Rightarrow$ $x_4 + x_5 = 30 - 8 = 22$.

So, cricket playing members $= x_1 + x_7 + x_4 + x_5 = 25 + 10 + 22$
$= 57$.

Q.9. Solution:- $u > v$ & $u_1 = v_1$, $u_2 = v_2$, $\vdots$, $u_K = v_K$ but $u_{K+1} > v_{K+1}$

Again $w > z$ & $w_1 = z_1$, $w_2 = z_2$, $\vdots$, $w_K > z_K$ but $w_{K+1} = z_{K+1}$.

So, $u_K + w_K = v_K + z_K$

$\Rightarrow (u+w)_K = (v+z)_K$

but $u_{K+1} + w_{K+1} > v_{K+1} + z_{K+1}$.

So, $u + w > v + z$.

Q.10. Solution:- (i) For every even number $n \geq 2000$, $a_n \leq 1$ and for every odd positive integer, $a_n \leq 1$.

Here, $\{a_n\}$ is a decreasing sequence and

$$a_{2000} = 0.9 + \frac{200}{2000} = 1.$$

$\therefore$ a_n satisfies Property P.

(ii) Let $n = 4K$ (even)

i.e., $$a_{4K} = 1 + \frac{1}{4K}\cos\left(\frac{4K\pi}{2}\right)$$

$$= 1 + \frac{1}{4K}\cos(2K\pi)$$

$$= 1 + \frac{1}{4K} > 1.$$

$[\because \cos(2K\pi) = 1]$ So, here a_n doesnot satisfy P.

Q.11. Solution:- $$P = 1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

If P is prime, then $x - 1 = 1 \Rightarrow x = 2$.

$\therefore$ $P = \dfrac{2^n - 1}{2 - 1} = 2^n - 1$ is a prime.

Let n is not a prime, then $n = pq$ [p, q are +ve integers]

So, $2^n - 1$ is divided by both $2^p - 1$ and $2^q - 1$

i.e., $2^n - 1$ is not a prime.

But we know $2^n - 1$ is prime, so, by contradiction n is also prime.

Q.12. Solution:- $x_n = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots\cdots \frac{2n-1}{2n}$

$$= \frac{1.2.3.4.\ldots\ldots(2n-1)(2n)}{(2.2\ldots\ldots \text{to } 2n \text{ times})(1.2.3\ldots\ldots n)^2}$$

$$= \frac{(2n)!}{2^{2n}.(n!)^2}$$

We need to show that $\frac{(2n)!}{2^{2n}.(n!)^2} \leq \frac{1}{\sqrt{3n+1}} \ \forall\, n \in \mathbb{N}.$

By induction, $P(1)$ is true.
Let $P(m)$ is true, i.e., $\frac{(2m)!}{2^{2m}.(m!)^2} \leq \frac{1}{\sqrt{3m+1}}$,

$$P(m+1) = \frac{(2m+2)!}{2^{2m+2}\,[(m+1)!]^2}$$

$$= \frac{(2m)!\,(2m+1)(2m+2)}{4.2^{2m}\,(m!)^2\,(m+1)^2}$$

$$= \frac{(2m)!}{2^{2m}.(m!)^2} \cdot \frac{(2m+1)}{2(m+1)} \leq \frac{1}{\sqrt{3m+1}} \cdot \frac{2m+1}{2(m+1)}.$$

$$\leq \frac{1}{\sqrt{3m+4}}\ ; \ [\text{Show it}]$$

$\therefore P(m+1)$ is true.
So, by induction $x_n \leq \frac{1}{\sqrt{3n+1}} \ \forall\, n \in \mathbb{N}.$

Q.13. Solution:-

(i) From $n! = \sum_{k=0}^{n} A_k x(x+1)(x+2)\cdots\cdots(x+k-1)(x+k+1)\cdots(x+n)$

Putting $x = -k$,

$$n! = \sum_{k=0}^{n} (-1)^{-k} A_k \cdot k!\,(n-k)!$$

$$\Rightarrow A_k = (-1)^k \binom{n}{k}.$$

(ii) Considering $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$.

Integrating w.r.t. x, we have

$$\frac{(1+x)^{n+1}}{n+1} = {}^nC_0 x + \frac{{}^nC_1 x^2}{2} + \dots\dots + \frac{{}^nC_n x^{n+1}}{n+1} + \text{constant}$$

Putting $x=0$, then constant $= \frac{1}{n+1}$.

So, $$\frac{(1+x)^{n+1}}{n+1} = {}^nC_0 x + \frac{{}^nC_1 x^2}{2} + \dots\dots + \frac{{}^nC_n x^{n+1}}{n+1} + \frac{1}{n+1}.$$

Again integrating w.r.t. x,

$$\frac{(1+x)^{n+2}}{(n+1)(n+2)} = \frac{{}^nC_0 x^2}{1.2} + \frac{{}^nC_1 x^3}{2.3} + \dots\dots + \frac{{}^nC_n x^{n+2}}{(n+1)(n+2)} + \frac{x}{n+1} + \text{constant}.$$

Putting $x=0$, then constant $= \frac{1}{(n+1)(n+2)}$.

So, $$\frac{(1+x)^{n+2}}{(n+1)(n+2)} = \frac{{}^nC_0 x^2}{1.2} + \frac{{}^nC_1 x^3}{2.3} + \dots\dots + \frac{{}^nC_n x^{n+2}}{(n+1)(n+2)} + \frac{x}{(n+1)} + \frac{1}{(n+1)(n+2)}.$$

Putting $x=-1$, we get

$$0 = \frac{{}^nC_0}{1.2} - \frac{{}^nC_1}{2.3} + \frac{{}^nC_2}{3.4} \dots\dots + (-1)^n \frac{{}^nC_n}{(n+1)(n+2)} - \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)}.$$

So, $$\binom{n}{0}\frac{1}{1.2} - \binom{n}{1}\frac{1}{2.3} + \binom{n}{2}\frac{1}{3.4} + \dots + (-1)^n \binom{n}{n}\frac{1}{(n+1)(n+2)}$$

$$= \frac{1}{n+2}. \quad \text{(Proved)}$$

Q.14. Hints:-

$$t_n = \frac{n+2}{n(n+1)(n+3)}$$

$$= \frac{1}{6}\left[\frac{1}{n+1} - \frac{1}{n+2}\right] + \frac{1}{6}\left[\frac{1}{n+2} - \frac{1}{n+3}\right] + \frac{2}{3}\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

So, $S = \sum_{n=1}^{n} T_n = \frac{1}{6}\sum_{n=1}^{n}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \frac{1}{6}\sum_{n=1}^{n}\left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \frac{2}{3}\sum_{n=1}^{n}\left(\frac{1}{n} - \frac{1}{n+1}\right).$

$$= \frac{1}{6}\left(\frac{1}{2} - \frac{1}{n+1}\right) + \frac{1}{6}\left(\frac{1}{3} - \frac{1}{n+3}\right) + \frac{2}{3}\left(1 - \frac{1}{n+1}\right)$$

$$= \frac{1}{6}\left[\frac{29}{6} - \frac{4}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right] \text{ (Proved)}$$

Q.15. Solution:-

We need to count the no. of solutions of $x_1 + x_2 + \cdots + x_8 = 7$, which satisfies $0 \le x_i \le 7$, $i = 1, 2, 3, \ldots, 8$. ------(1)

The number of solution of (1) is = coefficient of x^7 in $(1 + x + x^2 + \cdots + x^7)^8$.

= coefficient of x^7 in $(1 - x^8)^8 (1 - x)^{-8}$

= coefficient of x^7 in $(1 - 8x^8)(1 + {}^8C_1 x + {}^4C_2 x^2 + {}^{10}C_3 K + \cdots)$

$= {}^{14}C_7 = {}^{7+8-1}C_{8-1}$.

$= 3432$. (Ans)

Q.16.

We can prove it by induction.

For $n=1$, $K!$ divides $K!$.

Let this is true for $n=m$, i.e., $(K!)^m$ divides $(Km)!$.

For $n=m+1$,

$$\{K(m+1)\}! = (Km+K)!$$
$$= (Km+K)(Km+K-1)\cdots\{Km+K-(K-1)\}(Km)!$$

Now, $(K!)^m$ divides $(Km)!$ (by assumption)

So, we need to show $K!$ divides $(Km+K)(Km+K-1)\cdots\{Km+K-(K-1)\}$

This is multiplication of K consecutive positive integers and that is always divisible by $K!$.

Proved.

Q.17.

Now, discriminant $= b^2-4ac$ and it has to be a perfect square for the roots to be rational.

Let $b^2-4ac = p^2$

As, all a, b, c are odd implies p is odd.

Dividing the equation by 8 we get,

$1-4 \equiv 1 \pmod 8$ (As any odd number square congruent to 1 modulus 8)

$\Rightarrow -3 \equiv 1 \pmod 8$

Which is impossible.

$\Rightarrow$ The roots can't be rational.

Q.20. Solution:-

(ii) Now, $6u_n - 4u_{n-1} = 6(3+\sqrt{5})^n + 6(3-\sqrt{5})^n - 4(3+\sqrt{5})^{n-1} - 4(3-\sqrt{5})^{n-1}$

$= (3+\sqrt{5})^{n-1}(18+6\sqrt{5}-4) + (3-\sqrt{5})^{n-1}(18-6\sqrt{5}-4)$

$= (3+\sqrt{5})^{n-1}(3+\sqrt{5})^2 + (3-\sqrt{5})^{n-1}(3-\sqrt{5})^2$

$= (3+\sqrt{5})^{n+1} + (3-\sqrt{5})^{n+1}$

$= u_{n+1}$.

(i) Now, $u_1 = 3+\sqrt{5}+3-\sqrt{5} = 6 =$ integer.

$u_2 = (3+\sqrt{5})^2 + (3-\sqrt{5})^2 = 2(9+5) = 28 =$ integer.

Using (ii) and induction, u_n is always an integer.

(iii) u_1 is divisible by 2 and u_2 is divisible by 2^2.

Let for $n = K$ and $K+1$, u_K and u_{K+1} are divisible by 2^K and 2^{K+1} respectively.

Now, $u_{K+2} = 6u_{K+1} - 4u_K = 6 \times 2^{K+1} \times m_1 - 4 \times 2^K \times m_2$

$= 2^{K+2}(3m_1 - m_2)$.

$\Rightarrow u_{K+2}$ is divisible by 2^{K+2}.

Hence, by induction 2^n divides u_n.

Q.22. Let $P(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \frac{\text{odd number}}{\text{even number}}$ (to prove)

For $n = 2$, LHS $= P(2) = 1 + \frac{1}{2} = \frac{3}{2} = \frac{\text{odd number}}{\text{even number}}$ = not an integer.

Let us consider $P(m) = 1 + \frac{1}{2} + \cdots + \frac{1}{m} = \frac{\text{odd number}}{\text{even number}}$, = RHS.

For $P(m+1)$ LHS $= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} + \frac{1}{m+1}$

$$= \frac{K}{r} + \frac{1}{m+1}.$$

Case-I:- m is even, $\Rightarrow m+1$ is odd.

$$\frac{K}{r} + \frac{1}{m+1} = \frac{K(m+1)+r}{r(m+1)} = \frac{\text{odd number}}{\text{even number}} = \text{not an integer.}$$

Case-II:- m is odd. $\Rightarrow m+1$ is even.

(a) When $m+1 = 2^{\alpha}$ where $\alpha \in \mathbb{N}$.

In this case $r = 2^{\beta}.q$ where $1 \le \beta \le \alpha - 1$.

and q is an odd number.

$$\therefore \frac{K}{r} + \frac{1}{m+1} = \frac{K}{2^{\beta}q} + \frac{1}{2^{\alpha}} = \frac{K.\,2^{\alpha-\beta} + q}{2^{\alpha}.q}$$

$$= \frac{\text{even} + \text{odd}}{\text{even}}$$

$$= \frac{\text{odd number}}{\text{even number}}$$

(b) When $m+1 \neq 2^{\alpha}$.

$m+1 = 2^{s}.r$ where r is odd.

$\therefore r = 2^{\beta}.q$ where $s < \beta$ and q is odd.

$$\therefore \frac{K}{r} + \frac{1}{m+1} = \frac{K}{2^{\beta}q} + \frac{1}{2^{s}.r} = \frac{Kr + 2^{\beta-s}}{2^{\beta}.q.r}$$

$$= \frac{\text{odd number}}{\text{even number}}$$

Q.23.

Let $E = (1+\sqrt{2})^n + (1-\sqrt{2})^n$

Let $(1+\sqrt{2})^n = I + f$ where I is integer part and $0 \le f < 1$ is fractional part.

So, $I = [(1+\sqrt{2})^n]$

Also, $\sqrt{2} - 1 = \dfrac{1}{\sqrt{2}+1}$ $\quad [\because 0 < \sqrt{2} - 1 < 1]$

$\Rightarrow 0 < (\sqrt{2}-1)^n < 1.$

Let, $F = (\sqrt{2}-1)^n$

$\Rightarrow 0 < F < 1$

Case I:- When n is odd.

$I + f - F = (\sqrt{2}+1)^n + (\sqrt{2}-1)^n = 2\left[{}^nC_0 + {}^nC_2(\sqrt{2})^2 + \cdots\right]$

$= \text{even} = 2K \text{ (say)}$

$f - F = 2K - I =$ an integer.

Since $f - F$ is an integer, $f - F = 0$.

$\Rightarrow f = F$

$\Rightarrow I - 2K = 0$

$\Rightarrow I = 2K = [(1+\sqrt{2})^n] =$ even when n is odd.

Case II:- When n is even.

$I + f + F = (\sqrt{2}+1)^n + (\sqrt{2}-1)^n = 2K_1 =$ even number.

$\Rightarrow f + F = 2K_1 - I =$ integer.

Since $0 < f < 1$ and $0 < F < 1$

$\Rightarrow 0 < f + F < 2 \Rightarrow f + F = 1.$

$\therefore 2K_1 - I = 1 \Rightarrow I = 2K_1 + 1$

$\Rightarrow [(1+\sqrt{2})^n]$ is an odd integer.

Q.24. Solution:- As $3n+1$ is a perfect square, so let

$$3n+1 = a^2$$

$\Rightarrow$ a is not a multiple of 3.

$\Rightarrow$ a may be of the form either $3k+1$ or $3k+2$, $k \in I$.

Taking $a = 3k+1$, $3n+1 = (3k+1)^2 = 9k^2+6k+1$;

or, $n = 3k^2+2k$

or, $n+1 = 3k^2+2k+1 = k^2+k^2+(k+1)^2$;

i.e., sum of three perfect squares.

Taking $a = 3k+2$, $3n+1 = (3k+2)^2 = 9k^2+6k+4$;

or, $n = 3k^2+4k+1$,

or, $n+1 = 3k^2+4k+2 = k^2+(k+1)^2+(k+1)^2$

i.e., sum of 3 perfect squares. [Proved].

Q.25. Solution:- When 'n' is perfect square, then $\sqrt{n}$ is an integer.

When 'n' is not a perfect square, then let $\sqrt{n}$ is a rational number $= \frac{p}{q}$, where $\gcd(p,q)=1$.

i.e., $n = \frac{p^2}{q^2}$

$\because$ p and q are relatively prime to each other, p^2 and q^2 should be relatively prime to each other and $\frac{p^2}{q^2}$ can't be an integer. Hence, $\sqrt{n}$ is not a rational number.

Q.26. Solution:-

$$2^{2n} - 3n - 1$$

$$= 4^n - 3n - 1$$

$$= (1+3)^n - 3n - 1$$

$$= \{1 + 3n + 9.{}^nC_2 + \cdots + 3^n\} - 3n - 1$$

$$= 9\left({}^nC_2 + 3.{}^nC_3 + \cdots + 3^{n-2}\right)$$

i.e., $2^{2n} - 3n - 1$ is divisible by 9 $\forall n \geq 1$.

Q.27. Solution:- The roots of the equation $x^2+px+q=0$ are

$$x = \frac{-p \pm \sqrt{p^2-4q}}{2}.$$

As roots are rational, hence D is a perfect square.

i.e., $p^2-4q=k^2$, where $k \in I$.

or, $p^2 = k^2+4q$.

Now, when p = even, k^2+4q = even.

$\Rightarrow k^2$ = even, so k = even.

$$\Rightarrow x = \frac{-\text{even} \pm \text{even}}{2} = \text{Integer} \quad \text{------(i)}$$

Again, when p = odd, k^2+4q = odd

$\Rightarrow k^2$ = odd, so k = odd.

$$\Rightarrow x = \frac{-\text{odd} \pm \text{odd}}{2} = \text{Integer} \quad \text{------(ii)}$$

Hence the proof is complete.

Q.29. Solution:-

$$3^{2n+1}+2^{n+2}$$
$$= 3.3^{2n}+4.2^n$$
$$= 3(2+7)^n+4.2^n$$
$$= 3\left[2^n+{}^nC_1.2^{n-1}.7+\cdots\cdots+7^n\right]+4.2^n$$
$$= 7.2^n+3.7.{}^nC_1.2^{n-1}+\cdots\cdots+3.7^n$$
$$= 7\left(2^n+3.{}^nC_1.2^{n-1}+\cdots\cdots+3.7^{n-1}\right)$$
$$\equiv \text{multiple of } 7.$$

Alt. Proof:- $P(1)$ is true, $P(2)$ is true.

$$P(m):\ 3^{2m+1}+2^{m+2} \equiv 7M.$$

$$P(m+1) = 3^{2m+3}+2^{m+3} = 9.3^{2m+1}+2.2^{m+2}$$
$$= 2.3^{2m+1}+7.3^{2m+1}+2.2^{m+2}$$
$$= 2\left[3^{2m+1}+2^{m+2}\right]+7.3^{2m+1}$$
$$= 2\times 7M+7.3^{2m+1} \quad \Big|\ 7.$$

So, $P(m+1)$ is true. By induction, $P(n)$ is divisible by 7.

Q.31.

Let n is even.

Then n has two middle terms viz. $n/2$ and $n/2+1$.

Now, $1^K + n^K \equiv 1^K + (-1)^K \pmod{n+1}$

$\equiv 0 \pmod{n+1}$ (as K is odd)

Similarly, $2^K + (n-1)^K, 3^K + (n-2)^K, \ldots, (n/2)^K + (n/2+1)^K \equiv 0 \pmod{n+1}$.

Therefore, $n+1$ divides the expression.

Now, $n/2$ divides n^K.

Now, there are $1^K + 2^K + \ldots + (n-1)^K$

Now, $1^K + (n-1)^K \equiv 1^K + (-1)^K \pmod{n/2}$

$\equiv 0 \pmod{n/2}$ (as K is odd)

Similarly, $2^K + (n-2)^K, 3^K + (n-3)^K, \ldots, (n/2-1)^K + (n/2+1)^K$

$\equiv 0 \pmod{n/2}$.

And middle term $(n/2)^K$ is divisible by $n/2$.

Therefore $n/2$ divides the expression.

So, $\frac{n(n+1)}{2}$ divides the expression when n is even.

Similarly, we can prove for n = odd.

Q.33.

We have, $\frac{x^2+y^2}{2} \geq \left(\frac{x+y}{2}\right)^2$.

$\Rightarrow x^2+y^2 \geq K^2/2$.

Now, $x = y = K/2$ is not possible as K is odd.

Therefore, minimum value is $\frac{(K^2+1)}{2}$ and it attains when

$x = \frac{K-1}{2}$ and $y = \frac{K+1}{2}$.

Q.39. Let us take 3 integers even. So, they have a common factor 2. One of the odd integer and one of the even integer may get divisible by 3.

Let us take the another odd integer. This is clearly relatively prime to the other four as 2 and 3 doesn't divide the integer and if 5 divides or any prime > 5 divides the integer then these four will not be divisible by that prime.

Q.38.

Let $p = 2 \times 3 \times 5m + r$ $(30 = 2 \times 3 \times 5)$

Now, r can't be any integer which have 2, 3 or 5 as factor because in that case p will not be prime.

For example if $r = 14$ then $p = 2(3 \times 5 \times m + 7)$ and p can be factored. Here is contradiction.

So, r must be prime. r can be composite number 7^2 as after 2, 3, 5 next prime is 7 and the lowest composite number generated by 7 is $7^2 = 49$ which is greater than 30 but r less than 30.

So, r is either prime or 1.

Q.28. $f(x)$ and $g(x)$ are two polynomials (quadratic) with rational coefficients.

Now, they have an irrational root and irrational roots appear in conjugate pairs. Thus both polynomial of degree 2 have both roots in common.

Thus, $f(x) = rg(x)$ where r is rational number. (if not $f(x) = g(x)$).

Q. 30. Solution:-

Take $n = 2k+1$,

for $n=3$, $3^5 - 3 = 240 \mid 80$.

Now, $P(n) = n^5 - n$

$$= (2k+1)^5 - (2k+1)$$

$$= \{(2k-1)+2\}^5 - (2k-1)+2$$

$$= (2k-1)^5 + {}^5C_1 (2k-1)^4 . 2 + {}^5C_2 (2k-1)^3 . 2^2 + {}^5C_3 (2k-1)^2 . 2^3 + {}^5C_4 . (2k-1) 2^4 + 2^5 - (2k-1) + 2$$

$$= \{(2k-1)^5 - (2k-1)\} + 10(2k-1)^4 + \frac{5.4}{2.1} \times 4(2k-1)^3 + \frac{5.4}{2.1} . 8(2k)^2 + 5 \times 16(2k-1) + 30$$

$$\equiv \text{Multiple of } 80 + 10(16k^4 - 24k^2 + 16k)$$

$$\equiv \text{Multiple of } 80 + 80(2k^4 - 3k^2 + 2k)$$

$\therefore$ $(2k+1)^5 - (2k+1)$ is divisible by 80.

$\therefore$ By induction method, for all odd integer n, $n^5 - n$ is divisible by 80.

Q. 35. (a) Let $n = 2a+1$

Solution:-

$$n^4 = (2a+1)^4$$

$$= (4a^2 + 4a + 1)^2$$

$$= [4a(a+1) + 1]^2$$

$$= 16a^2(a+1)^2 + 8a(a+1) + 1$$

$$= 16[a(a+1)]^2 + 8a(a+1) + 1.$$

Now, $a(a+1)$ is divisible by 2.

$$\therefore n^4 \equiv 1 \pmod{16}.$$

(b) <u>Solution:-</u> Now,

$$n_1^4 \equiv 1 \pmod{16}$$
$$n_2^4 \equiv 1 \pmod{16}$$
$$\vdots \qquad \vdots$$
$$n_8^4 \equiv 1 \pmod{16}$$

$$n_1^4 + n_2^4 + \cdots + n_8^4 \equiv 8 \pmod{16}$$

but $1993 \equiv 9 \pmod{16}$

So, the value of $n_1^4 + n_2^4 + \cdots + n_8^4$ can't be 1993.

Q.44. <u>Solution:-</u>

$2(n-2) > n-2 \Rightarrow 2n-4+2 > n \Rightarrow 2(n-1) > n \cdots\cdots$ (i)

$3(n-3) > n-3 \Rightarrow 3n-9 > n-3 \Rightarrow 3(n-2) > n \cdots\cdots$ (ii)

Similarly, we have $4(n-3) > n \cdots\cdots$ (iv)

$5(n-4) > n \cdots\cdots$ (v)

$\vdots$

Multiplying all these upto $(n-2)$ terms, we get

$$[1.2.3.\ldots(n-1)]^2 > n^{n-2}$$

$$\Rightarrow [(n-1)!]^2 > \frac{n^n}{n^2}$$

$$\Rightarrow [n!]^2 > n^n.$$

Q.46. <u>Solution:-</u> Number of onto functions from $\{1,2,3,\ldots\ldots,n\}$ to $\{1,2\}$ is $2^n - 2$.

Here, $A = \{1,2,3,\ldots\ldots,n\}$, $B = \{1,2,3\}$, for each $i \in A$ have 3 possibilities, so total no. of $f(n)$ from A to B is 3^n.

But there are $\binom{3}{2}(2^n-2)$ $f(n)$ image consist of 2 points and 3 $f(n)$ whose image is singleton. Hence, total number of onto functions $f(n)$ from A to B is $\left\{3^n - \binom{3}{2}(2^n-2) - 3\right\}$.

Q.40. (i)

Let p divides $1+ml$ and $1+(k+m)l$

So, $1+ml \equiv 0 \pmod p$

And, $1+(k+m)l \equiv 0 \pmod p$

$\Rightarrow kl \equiv 0 \pmod p$

$\Rightarrow$ p divides either k or l.

Let p divides l then p can't divide $1+ml$.

Let p divides k then p divides again l as k divides l.

Again p cannot divide $1+ml$.

Here is the contradiction.

$\Rightarrow$ They are relatively prime.

Q.42.

Clearly, $f(x) = (x-a_1)(x-a_2)(x-a_3)(x-a_4)Q(x) + 3$

where, $Q(x)$ is a polynomial.

Now, $f(b) = (b-a_1)(b-a_2)(b-a_3)(b-a_4)Q(b) + 3 = 14$

$\Rightarrow (b-a_1)(b-a_2)(b-a_3)(b-a_4)Q(b) = 11$

Now, 11 is a prime and 11 is factor of at least 4 distinct integers. As a_1, a_2, a_3, a_4 are distinct hence $b-a_1$, $b-a_2$, $b-a_3$, $b-a_4$ are distinct.

But 11 can have maximum 3 factors viz. $11, 1, -1$ or $-11, 1, -1$.

But it can't have 4 factors.

So, $f(b) = 14$ is not possible.

Q.45.

To satisfy first equation, $3 \odot x = 4$ and $2 \odot y = 3$, or,
$3 \odot x = 3$ and $2 \odot y = 4$.

First case, $x = 3$, $y = 4$ but it doesn't satisfy second equation.
Second case, $x = 1$, $y = 2$ but it doesn't satisfy second equation.

Now, $3 \odot x = 0$ and $2 \odot y = 2$ or $3 \odot x = 2$ and $2 \odot y = 0$.
First case, $x = 0$, $y = 1$ but it doesn't satisfy second equation.
Second case, $x = 4$, $y = 0$ but it doesn't satisfy second equation.
So, no solution exists.

Q.37. Let $\frac{p-1}{4} = p_1$ and $\frac{p+1}{2} = p_2$

$\Rightarrow p^2 - 1 = 4p_1 \times 2p_2$

Now, $(p-1)p(p+1)$ is divisible by 3.

Also, $(p-1)$ and $(p+1)$ are even.

Thus $p^2 - 1$ is divisible by 12.

$\therefore 12 \mid 8p_1p_2$

$\Rightarrow 3 \mid 2p_1p_2$

$\therefore p_1$ or p_2 is 3 ($\because$ they are prime)

Now, if $p_1 = 3$, $p = 13$.

Q.48. Solution :- Label the children 1, 2, 3. Let A_i be the ways to distribute gifts so that child i gets no gift. Let B denotes the way to distribute gifts so that every child gets a gift.

For every way of giving out gifts, either every child got a gift or (at least) one child did not get any gifts,

hence $3^5 = |B| + |A_1 \cup A_2 \cup A_3|$

$$\therefore\ 3^5 = |B| + |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

[Now, $|A_1| = 2^5$ (because A_1 is 1st child doesn't get a gift) $|A_2| = |A_3| = 2^5$. Now, $|A_1 \cap A_2| = |A_2 \cap A_3| = |A_1 \cap A_3| = 1^5$; $|A_1 \cap A_2 \cap A_3| = 0$

So, $|B| = 3^5 - 3 \times 2^5 + 3 \times 1^5 - 0 = 243 - 96 + 3 = 150$.

So, total number of ways = 150.

Q.49. Solution:- Clearly, number of possible arrangement is

$$\frac{(x+y+z)!}{x!\,y!\,z!} = \frac{30!}{x!\,y!\,z!}$$

This expression is maximum when $x!\,y!\,z!$ = minimum.

That is posible when $x = y = z$ as given that $x + y + z = 30$.

So, $x = y = z = 10$.

Q. 50. Solution:- Words starting with 'a' $\rightarrow 4!$
" " " 'b' $\rightarrow 4!$
" " " 'c' $\rightarrow 4!$
" " " 'd' $\rightarrow 3!+3!+3!+3$

(i.e., $da \rightarrow 3!$, $db \rightarrow 3!$, $dc \rightarrow 3!$, $de \rightarrow 3$)

Total no. of words before debac including it is =

$$3\times 4! + 3\times 3! + 3 = 93.$$

Q. 51. Solution:- (a) Put 1 ball in each box, so, we have $m-x$ identical balls to be distributed in K boxes.

Let, the jth box got i_j balls out of $(m-K)$ balls, where $j = 1,2,3,\ldots,K$; $0 \le i_j \le m-K$.

So, we need to find the no. of solutions of the equation $i_1+i_2+\cdots+i_K = m-K$, where each i_j is non-negative integers.

$\therefore$ Total number of solutions = coefficient of x^{m-1} in $(1-x)^{-K}$

$$= \binom{m-1}{K-1}.$$

(b) Hints:- Number of arrangements are $= \binom{n}{m+1}\binom{m-1}{K-1}$.

Q. 52. Solution:- (i) Let $S = \{n_1, n_2, \ldots, n_K\}$ be a set with K distinct elements. Given $n_i \geq 0$ and $\sum_{i=1}^{K} n_i = 100$.

So, total number of all possible ordered K-tuples of such kinds are $= \binom{K-1+100}{100}$.

(ii) Here $K = 4$, and $\sum_{i=1}^{4} n_i \le 100$

So, no. of such possible cases are $= \binom{4-1+101}{4}$

$$= \binom{104}{4}.$$

Q.55.

(i) We can choose 1 element from n elements in ${}^nC_1$ ways.
We can choose 2 elements from n elements in ${}^nC_2$ ways.

. . . . . .

. . . . . .

We can choose $n-1$ elements from n elements in ${}^nC_{n-1}$ ways.

We are choosing for one subset and as it is nonempty so ${}^nC_0$ and ${}^nC_{n-1}$ will not come into picture.

So, total number of ways $= {}^nC_1 + {}^nC_2 + \cdots + {}^nC_{n-1} = 2^n - 2$.

Therefore, $S_2^n = 2^n - 2$.

(ii) Now,

Case I:-

S_K^{n+1}: To get this partition either putting $(n+1)^{th}$ elements in one of the existing K classes with n elements.

We can choose 1-class out of the K classes in ${}^KC_1 = K$ ways and number of partition for class is S_K^n ways.

So, total number of partition is KS_K^n.

Case II:- Here

S_K^{n+1}: To get this partition, we can form a new class and appending that class to existing $K-1$ classes.

We can partition n elements into $(K-1)$ classes in S_{K-1}^n ways and form a new class (K^{th} class) with $(n+1)^{th}$ element.

$$\therefore \quad S_K^{n+1} = S_{K-1}^n + KS_K^n.$$

Q.53. Solution:- We need to select odd number of odd numbers. We can select any number of even number.

So, number of odd numbers below 50 = 25.

And, number of even number $\leq 50 = 25$.

We can select any 1 odd number in $\binom{25}{1}$ ways.

We can select any 3 odd number in $\binom{25}{3}$ ways.

. . . . . .

. . . . . .

We can select any 25 odd numbers in $\binom{25}{25}$ ways.

So, total number of ways to select odd numbers

$$= \binom{25}{1} + \binom{25}{3} + \cdots\cdots + \binom{25}{25} = 2^{24}.$$

Now, we can select even number in

$$\binom{25}{0} + \binom{25}{1} + \binom{25}{2} + \cdots + \binom{25}{25} = 2^{25} \text{ ways.}$$

So, total number of ways $= 2^{24} \times 2^{25} = 2^{49}$.

Q.56. Solution:-

$\boxed{1}$ (1) $\boxed{2}$ (2) $\boxed{3}$ (3) $\boxed{4}$ (4) $\boxed{5}$ .....

We choose 4 integers as shown in above by circle.

So, there are maximum 5 spaces between them shown in figure by boxes.

Now, let us say, 2, 3, 4 spaces, i.e., boxes are to be filled by another $n-4$ integers (4 integers already chosen for 4 circles).

$\therefore$ Number of ways $= {}^{n-5}C_2$ (As number of ways are $\binom{n-1}{r-1}$ for at least one to be there).

Similarly, for 1,2,3,4 boxes and 2,3,4,5 boxes to be filled by other $n-4$ integers, number of ways $= 2 \times \binom{n-5}{3}$.

For 5 boxes to be filled by other $n-4$ integers, number of ways $= \binom{n-5}{4}$.

$\therefore$ Total number of ways $= {}^{n-5}C_2 + 2 \times {}^{n-5}C_3 + {}^{n-5}C_4$

$= {}^{n-4}C_3 + {}^{n-4}C_4$

$= {}^{n-3}C_4$.

Q.57. Solution:-

Let x_1 number of A, x_2 number of B and x_3 number of C are chosen where $x_1, x_2, x_3 > 0$.

Now, $x_1 + x_2 + x_3 = 6$.

Number of positive solution of this equation is ${}^{6-1}C_{3-1} = {}^5C_2 = 10$.

So, combinations are as follows:

4A, 1 B, 1C, number of words $= \frac{6!}{4!} = 30$.

3A, 2B, 1C, number of words $= \frac{6!}{3!\,2!} = 60$.

3A, 1 B, 2C, number of words $= \frac{6!}{3!\,2!} = 60$.

2A, 1 B, 3 C, number of words $= \frac{6!}{2!\,3!} = 60$.

2A, 2B, 2 C, " " " $= \frac{6!}{2!\,2!\,2!} = 90$.

2A, 3 B, 1 C, " " " $= \frac{6!}{2!\,3!} = 60$.

1A, 1B, 4C, " " " $= \frac{6!}{4!} = 30$.

1 A, 2B, 3C, " " " $= \frac{6!}{2!\,3!} = 60$.

1 A, 3B, 2C, " " " $= \frac{6!}{3!\,2!} = 60$.

1 A, 4B, 1 C, " " " $= \frac{6!}{4!} = 30$.

Total number of words $= 30 \times 3 + 60 \times 6 + 90 = 540$.

Q.58. Solution:-

If $K \leq 2$, then "NO" $\Rightarrow$ Sure winning.

If $K = 1$, then it is obvious.

If $K = 2$ any of the two kinds of ball must be more than 10 in numbers.

Hence at least one of the 10 groups of 3 balls must contain 2 similar type of balls.

Where as, if $K = 3$ then if each type of balls are 10 in numbers then "NO" may not be sure wining.

If $K \geq 21$, then "YES" $\Rightarrow$ Sure winning.

Because let if $K = 21$, if a single type of ball is 11 in number resulting in duplicate type of ball in any group, total ball is $(30-11) = 19$ and remaining types and $(21-1) = 20$ and it is not possible.

Let $K = 20$, we can have 11 balls of single kind and 19 balls of different kinds $\Rightarrow$ not sure winning.

Q.73. $x^3 + Gx + H = 0$

Let the roots be $\alpha + i\beta$, $\alpha - i\beta$ and γ.

Sum of the roots are zero.

i.e., $2\alpha + \gamma = 0$.

As γ is real,

Now, $(\alpha + i\beta)(\alpha - i\beta)\gamma = H/1$

$\therefore (\alpha^2 + \beta^2)\gamma = H$.

$\therefore$ H is real.

Also, $(\alpha + i\beta)(\alpha - i\beta) + (\alpha + i\beta)\gamma + (\alpha - i\beta)\gamma = G$.

$\Rightarrow \alpha^2 + \beta^2 + \alpha\gamma + i\beta\gamma + \alpha\gamma - i\beta\gamma = G$

$\Rightarrow \alpha^2 + \beta^2 + \alpha\gamma + \alpha\gamma = G$

$\therefore$ G is real.

Q.59. Solution :- The number of squares with sides of unit length $= n \times n = n^2$.

The number of squares with sides of length 2 units is $= (n-1) \times (n-1) = (n-1)^2$ and so on.

$\therefore$ Total number of squares $= n^2 + (n-1)^2 + \dots + 2^2 + 1^2$

$$= \frac{n(n+1)(2n+1)}{6}.$$

Q. 60. Solution:- If 'a' is a member of some equivalence class then it's distinct digit determine the equivalence class completing. Hence no. of equivalence classes are the number of ways in which 'i' integers can be selected from $\{1,2,3,\dots,9\}$ for $2 \le i \le 5$ and $\{1,2,3,\dots,9\}$ for $i=1$.

Now, this can be done in $9 + \sum_{i=2}^{5} \binom{10}{i} = \sum_{i=1}^{5} \binom{10}{i} - 1$.

Q. 61. Solution:- $6x^2 - 25x + 12 + \frac{25}{x} + \frac{6}{x^2} = 0$;

$$\Rightarrow 6x^2 + 12 + \frac{6}{x^2} - 25x + \frac{25}{x} = 0$$

$$\Rightarrow 6x^2 - 12 + \frac{6}{x^2} - 25x + \frac{25}{x} + 24 = 0$$

$$\Rightarrow 6\left(x - \frac{1}{x}\right)^2 - 25\left(x - \frac{1}{x}\right) + 24 = 0$$

let, $x - \frac{1}{x} = y$; so, $6y^2 - 25y + 24 = 0$

$$\Rightarrow y = \frac{25 \pm \sqrt{625 - 576}}{2 \times 6} = \frac{8}{3} \text{ or } \frac{3}{2}.$$

When $y = \frac{8}{3}$, $x = 3$ or $\frac{1}{3}$,

for $y = \frac{3}{2}$, $x = 2$ or $-\frac{1}{2}$.

Q. 62. Solution:- $\Delta = \begin{vmatrix} 1 & 1 \\ a & 1 \end{vmatrix} = 1 - a$; $\Delta_1 = \begin{vmatrix} 2 & 1 \\ b & a \end{vmatrix} = 2a - b$

$$\Delta_2 = \begin{vmatrix} 1 & 2 \\ a & b \end{vmatrix} = b - 2a.$$

(i) For exactly one solution, $\Delta \neq 0$ i.e. $1 - a \neq 0 \Rightarrow a \neq 1$.

(ii) For no solution, $\Delta = 0$, i.e. $a = 1$, $\Delta_1 \neq 0$, $\Delta_2 \neq 0$. i.e., $2a \neq b$.

(iii) For more than one solution, $\Delta = \Delta_1 = \Delta_2 = 0$, $a = 1$, $b = 2$.

Q.63. Solution:- Let 1st two lines intersect.

$$ax + by = a + b \quad \text{——— (i)} \times b$$
$$bx - (a+b)y = -a \quad \text{——— (ii)} \times a$$

$$abx + b^2 y = ab + b^2$$
$$abx - a(a+b)y = -a^2$$
$$- \quad + \qquad\qquad +$$

$$y = 1.$$

$$\therefore x = \frac{a+b-by}{a} = 1.$$

In the third line, $(a+b)x - ay = LHS = b = RHS$.

So, three straight lines are concurrent.

Q.64. Solution:- $2f(x) + 3f(-x) = 15 - 4x$

Put $x = -x$, $2f(-x) + 3f(x) = 15 + 4x$

Solving, we get, $f(x) = 3 + 4x$

Q.65. Solution:- Let $P = ax^2 + bx + c$

$$= \frac{1}{4a}\left[4a^2x^2 + 4abx + 4ac\right]$$
$$= \frac{1}{4a}\left[(2ax)^2 + 4axb + b^2\right] + \frac{1}{4a}\left[4ac - b^2\right]$$
$$= \frac{(2ax+b)^2}{4a} + \frac{1}{4a}\left[4ac - b^2\right]$$

P is minimum when $(2ax+b) = 0$, i.e. $x = -\frac{b}{2a}$,

and $P_{min} = \frac{4ac - b^2}{4a}$.

Q.66. Solution:- $y = \frac{x^2 + 2x + c}{x^2 + 4x + 3c}$;

$\Rightarrow x^2y + 4xy + 3cy = x^2 + 2x + c$

$\Rightarrow (y-1)x^2 + 2x(2y-1) + c(3y-1) = 0$ [$\because x$ is real]

$\therefore \{2(2y-1)\}^2 - 4(y-1).c(3y-1) \geq 0$

$\Rightarrow c \leq \frac{(2y-1)^2}{(y-1)(3y-1)}$ $\quad \because 0 < c < 1$,

So, $\frac{1}{3} < y < 1$.

Q.67.

$$2\log_{2x+3} x < 1$$

Let, $2x+3 > 0$

$\Rightarrow x > -\frac{3}{2}$.

Therefore, $x^2 < 2x+3$

$\Rightarrow x^2 - 2x - 3 < 0$

$\Rightarrow (x-1)^2 < 4$

$\Rightarrow |x-1| < 2$

$\Rightarrow -2 < x-1 < 2$

$\Rightarrow -1 < x < 3$.

So, $-1 < x < 3$ is a solution.

Now, let $2x+3 < 0$, i.e., $x < -3/2$

Then, $x^2 > 2x+3$

$\Rightarrow (x-1)^2 > 4$

$\Rightarrow$ Either $x-1 > 2$ or $x-1 < -2$

$\Rightarrow x > 3$ or $x < -1$

$\Rightarrow x < -3/2$ is another solution.

Q. 68.

(i) Let the roots are $p-d, p, p+d, p+2d$.

Sum of the roots $= 4p+2d = 0$

$\Rightarrow d = -2p$.

So, roots are $3p, p, -p, -3p$.

Product of roots $= 9p^4 = m^2 \Rightarrow m = 3p^2$.

Sum of the roots taken two at a time $= 3p^2 - 3p^2 - 9p^2 - p^2 - 3p^2 + 3p^2$

$= -10p^2$

$= -(3m+2)$.

$\Rightarrow 10p^2 = 9p^2 + 2$

$\Rightarrow p^2 = 2$

$\Rightarrow m = 3p^2 = 6$.

(ii) Let x_1, x_2 are the roots of the equation.

$\therefore -b = x_1 x_2$

$\Rightarrow |b| = |x_1||x_2| < 1 \times 1 = 1$.

Roots are $= \left\{a \pm \sqrt{a^2+4b}\right\}/2$.

Now, $\dfrac{a+\sqrt{a^2+4b}}{2} < 1$.

$\Rightarrow a^2+4b < 4-4a+a^2$

$\Rightarrow a+b < 1$.

Also, $-1 < \dfrac{a-\sqrt{a^2+4b}}{2}$

$\Rightarrow 2+a > \sqrt{a^2+4b}$

$\Rightarrow 4+4a+a^2 > a^2+4b$

$\Rightarrow b-a < 1$.

Q.69.

Roots of $ax^2 - 2bx + c = 0$ are $\dfrac{b \pm \sqrt{b^2 - ac}}{a}$.

Similarly for other two equations, roots are $\dfrac{c \pm \sqrt{c^2 - ab}}{b}$ and $\dfrac{a \pm \sqrt{b^2 - bc}}{c}$.

All roots are positive.

$b^2 - ac \geq 0$, $c^2 - ab \geq 0$, $a^2 - bc \geq 0$.

$\Rightarrow a^2 b^2 c^2 \geq a^2 b^2 c^2$.

$\Rightarrow$ '=' holds, i.e., $b^2 = ac$, $c^2 = ab$, $a^2 = bc$.

$\Rightarrow a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\Rightarrow \frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] = 0$.

$\Rightarrow a = b = c$.

Q.70.

$x^4 - 4x^3 + ax^2 + bx + 1 = 0$

Let the roots be $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 4$

$\alpha_1 \alpha_2 \alpha_3 \alpha_4 = 1$.

AM $\geq$ GM $\Rightarrow \dfrac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{4} \geq \sqrt[4]{\alpha_1 \alpha_2 \alpha_3 \alpha_4}$.

$\Rightarrow \dfrac{4}{4} \geq \sqrt[4]{1}$

$\Rightarrow 1 \geq 1$

'=' holds. $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$.

Q.71.

Now, $x - 2y - xy^2 = 0$

$\Rightarrow x = \dfrac{2y}{1-y^2}$ (x takes real values when $y \neq \pm 1$)

Putting this value in the second equation, we get,

$$\frac{2y}{1-y^2} + y + ay \times \frac{2y}{(1-y^2)} = a$$

$$\Rightarrow y^3 - 3ay^2 - 3y + a = 0.$$

As a is real, so all coefficients of the equation are real.
Hence complex roots will come in pair, i.e., if z is a root then z' (conjugate) is also a root.
And the equation is 3 degree (i.e., odd)
So, there is at least one real root.

So, y attains real root.

If $y = \pm 1$ then $a = \pm 1$.

So, the equation attains real values of x excluding $y = \pm 1$ if $a \neq \pm 1$.

$\Rightarrow x$ also takes real values.

So, the equation attains real solution in x and y except $a = \pm 1$.

Q.68. Solution:- (i) $x^4-(3m+1)x^2+m^2=0$

Let four roots be $\alpha-3\beta,\ \alpha-\beta,\ \alpha+\beta,\ \alpha+3\beta$.

So, Sum of roots = coefficient of x $=0$

$\therefore\ \alpha=0$.

So, roots are $-3\beta, -\beta, \beta, 3\beta$.

$\therefore\ -3\beta^2+3\beta^2-9\beta^2-\beta^2-3\beta^2+3\beta^2=-(3m+1)$

$\Rightarrow\ -10\beta^2=-(3m+1)$

$\therefore\ \beta^2=\dfrac{3m+1}{10}$.

Also, $9\beta^4=m^2$

$\therefore\ 9\left(\dfrac{3m+1}{10}\right)^2=m^2$

$\Rightarrow\ 9m+3=\pm 10m \Rightarrow m=3, -\frac{3}{19}$.

(ii) $x^2-ax-b=0$

Let roots be α, β, $|\alpha|<1$, $|\beta|<1$,

$\therefore\ |\alpha+\beta|\le|\alpha|+|\beta|<2$, as $|\alpha|<2$, or, $|\alpha||\beta|<1$

or, $|\alpha\beta|<1$ $\therefore\ |b|<1$.

Again, $ab<2$, $|b|-|a|<1$, and $|b|-|a|\le|b-a|$

$\therefore\ b-a<1$ and $b+a<1$.

Q.74. Solution:- Let the no. be (xyz), i.e. $N=100x+10y+z$

Given $x^2+y^2+z^2=146$, ... (i)

$4x=y+z$ ..... (ii)

$100z+10y+x=100x+10y+z+297$

$\Rightarrow\ z-x=3$ ..... (iii)

Solving (i), (ii) and (iii), we have $x=4, y=9, z=7$.

So, the number is 497.

Q.75. **Solution:-** Let $y = 1-\sqrt{x}$ .....(i)

and $y_0 = 3\sqrt{x}$ .....(ii)

For function (i) $x=0,\ y=1$; $x=1,\ y=0$; $x=\frac{1}{4},\ y=\frac{1}{2}$; $x=4,\ y=1$.

This is a continuous function curve which decreases from 1 to 0 in the interval $0 \le x \le 1$.

For function (ii) $x=0,\ y_0=0$; $x=1,\ y_0=1$; $x=\frac{1}{8},\ y_0=\frac{1}{2}$;

This is also a continuous function curve and it increases from 0 to 1 in the interval $0 \le x \le 1$.

Hence, they must meet each other, i.e., their value will be same at some point between $0 \le x \le 1$.

Hence, the given equation has only one real root.

Q.76. **Solution:-** $y = \dfrac{x^2-x}{1-mx}$

$$\Rightarrow y - mxy = x^2 - x$$

$$\Rightarrow x^2 + (my-1)x - y = 0$$

$$\therefore (my-1)^2 + 4y \geq 0 \quad [\because x \text{ is real}]$$

$$\Rightarrow m^2y^2 + (4-2m)y + 1 \geq 0$$

$$\therefore (4-2m)^2 - 4m^2 \geq 0 \quad [\because y \text{ is real}]$$

$$\Rightarrow (2-m)^2 - m^2 \geq 0$$

$$\Rightarrow 2-2m \geq 0 \Rightarrow m \le 1.$$

Q.77. **Solution:-** Let us take the sequence of numbers as

$$\{x^{n-1}, x^{n-2}, \ldots\ldots, x^0\}$$

Applying AM $\geq$ GM inequality :-

$$\frac{1}{n}\left(x^{n-1} + x^{n-2} + \cdots + x^0\right) \geq \sqrt[n]{x^{(n-1)+(n-2)+\cdots+0}}$$

or, $$\frac{1}{n}\cdot\frac{x^n-1}{x-1} \geq \left\{x^{\frac{n(n-1)}{2}}\right\}^{\frac{1}{n}}$$

or, $$\frac{x^n-1}{x-1} \geq nx^{\frac{n-1}{2}}.$$

Q.81.

Without loss of generality, let us assume that

$$a \geq b \geq c \geq d \geq e.$$

$$(a+b+c)^3 = 3e \quad ; \quad (c+d+e)^3 = 3b$$

$$\therefore \quad (a+b+c)^3 \geq (c+d+e)^3$$

$$\Rightarrow \quad 3e \geq 3b.$$

but our assumption says $3b \leq 3e$.

So, $3e = 3b$.

So, $a = b = c = d = e = 1/3$.

Q.79. $(1+\sin^2\theta_1)(1+\cos^2\theta_1) = 1+\sin^2\theta_1+\cos^2\theta_1+\sin^2\theta_1\cos^2\theta_1$

$$= 2 + \left(\frac{1}{4}\right)\sin^2 2\theta_1$$

Now, $\sin^2 2\theta_1 \leq 1$

$$\Rightarrow \left(\frac{1}{4}\right)\sin^2 2\theta_1 \leq \frac{1}{4}$$

So, $2 + \left(\frac{1}{4}\right)\sin^2 2\theta_1 \leq 2 + \frac{1}{4} = \frac{9}{4}$.

So, $(1+\sin^2\theta_1)(1+\cos^2\theta_1) \leq \frac{9}{4}$

Similarly others.

Hence, multiplying all such inequalities, we get the required inequality. Hence proved.

Q.80. Solution:- $b^2+c^2 \geq \frac{(b+c)^2}{2}$ $\left[\because 2(b^2+c^2) = (b+c)^2+(b-c)^2 \Rightarrow 2(b^2+c^2) \geq (b+c)^2\right]$

i.e., $\frac{b^2+c^2}{b+c} \geq \frac{b+c}{2}$

Similarly, $\frac{c^2+a^2}{c+a} \geq \frac{c+a}{2}$ & $\frac{a^2+b^2}{a+b} \geq \frac{a+b}{2}$.

Adding, we get $\frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} + \frac{a^2+b^2}{a+b} \geq a+b+c$.

Q.82. Solution:- $\frac{1+a}{2} \geq \sqrt{a}$, $\frac{1+b}{2} \geq \sqrt{b}$, $\frac{1+c}{2} \geq \sqrt{c}$, $\frac{1+d}{2} \geq \sqrt{d}$.

Multiplying corresponding sides of the above inequalities, we have,

$$(1+a)(1+b)(1+c)(1+d) \geq 16\sqrt{abcd} \geq 16.$$

Q.83. Solution:- Let $S = \left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2$

$$= a^2+b^2+\frac{(a^2+b^2)}{a^2b^2}+4$$

$$= S_1+S_2+4.$$

$S_1 = a^2+b^2 \geq \frac{1}{2}$ $\left[(a-b)^2 \geq 0 \Rightarrow a^2+b^2-2ab \geq 0 \Rightarrow (a+b)^2 \geq 4ab \Rightarrow 4ab \leq 1 \Rightarrow ab \leq \frac{1}{4}\right.$

$\left. a^2+b^2 \geq 1-2\left(\frac{1}{4}\right) = \frac{1}{2}\right]$

$S_2 = \frac{a^2+b^2}{a^2b^2}$, $a^2b^2 \leq \frac{1}{16}$

$\therefore \frac{1}{a^2b^2} \geq 16$.

$\therefore S_2 \geq 8$

$\therefore S_1+S_2 \geq \frac{1}{2}+8$

$\therefore S \geq \frac{1}{2}+8+4 = \frac{25}{2}$.

Q.85.

(i) For $n=1$, $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$.

True for $n=1$.

For $n=k$, $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$.

For $n=k+1$,

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k (\cos\theta + i\sin\theta)$$

$$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$

$$= \cos k\theta\cos\theta + i\sin k\theta\cos\theta + i\cos k\theta\sin\theta - \sin k\theta\sin\theta$$

$$= \cos(k\theta+\theta) - i\sin(k\theta+\theta)$$

$$= \cos(k+1)\theta - i\sin(k+1)\theta.$$

Hence proved.

(ii) Do yourself.

Q.84. Solution:- We know $\cos^2(x^3+x) \le 1$.

$$\frac{2^x+2^{-x}}{2} \geq \sqrt{2^x \cdot 2^{-x}}$$

$$\Rightarrow \quad 2^x+2^{-x} \geq 2$$

$$2\cos^2(x^3+x) = 2^x+2^{-x} \geq 2$$

$\therefore \cos^2(x^3+x) = 1$

For $x=0$, the equation is satisfied. [Since $x=\pi$ doesn't hold]

Q.86. Solution:- Let $z = x+iy$, then $\left|\frac{z+1}{z-1}\right| \le 1$,

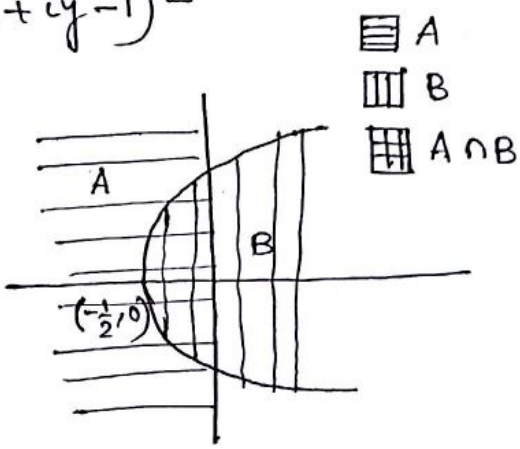
$\Rightarrow |z+1| \le |z-1|$,

$\Rightarrow (x+1+iy)^2 \le (x+iy-1)^2$

$\Rightarrow 4x \le 0$

$\Rightarrow x \le 0$.

$A = \{z : x \le 0\}$ ..... (i)



Also, $|z| - Re(z) \le 1$

$\Rightarrow \sqrt{x^2+y^2} \le x+1$

$\Rightarrow x^2+y^2 \le x^2+2x+1$

$\Rightarrow y^2 \le 2x+1 = 2(x+\frac{1}{2})$

$\therefore y^2 = 2(x+\frac{1}{2})$ is a parabola, having its vertex at $(-\frac{1}{2},0)$ and axis on X-axis.

Q.88. Solution:- Now, $z_1 = a+ib$, $z_2 = x+iy$, $0 = 0+i.0$

and let $z_1, z, 0$ be collinear then $\begin{vmatrix} 0 & 0 & 1 \\ a & b & 1 \\ x & y & 1 \end{vmatrix} = 0 \Rightarrow \frac{a}{b} = \frac{x}{y} = k$ (say)

$\therefore a = bk, x = yk$.

$\therefore z_1 = bk+ib = b(k+i)$, $z_2 = y(k+i)$.

So, $\exists$ some real 'r' or 's' such that $z = rz_1+sz_2 = rb(k+i)+sy(k+i)$

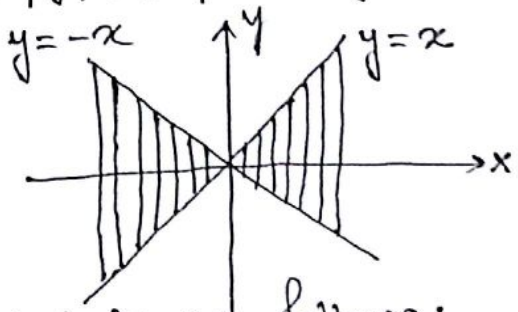
$= (rb+sy)(k+i)$,

which doesnot hold good.

So, z_1, z_2 and 0 should not be collinear.

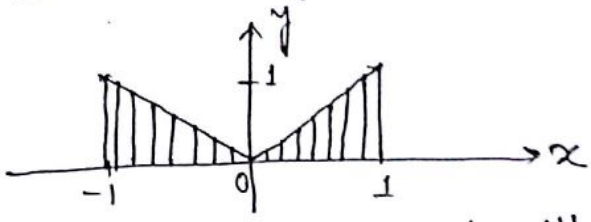
Q.90. Solution:- The graph $|y| \le |x|$ is as follows:

Shaded region represents $|y| \le |x|$.



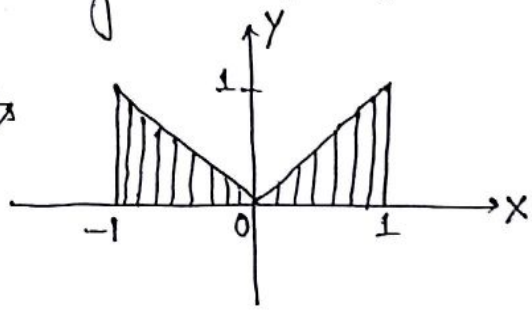
Now, the graph of $|x| \le 1$ is as follows:

Shaded region represents $|x| \le 1$.



So, the combined graph of $|y| \le |x| \le 1$ will be:

Shaded region represents $|y| \le |x| \le 1$.



Q.91. Solution:- $ax + by + c = 0$

$$\Rightarrow y = -\frac{ax+c}{b} > 0.$$

$$\therefore \frac{ax+c}{b} < 0 \Rightarrow \frac{acx^2+c^2}{bc} < 0.$$

There will be at least one point on the line for which $x > 0$ and $y > 0$ [$\because$ It passes through the 1st quadrant]

(i) If $acx + c^2 > 0$, then $bc < 0$, now, $x > 0$, $c^2 > 0$

$\therefore ac > 0.$

(ii) If $acx + c^2 < 0$, then $bc > 0$, now, $x > 0$, $c^2 > 0$.

$\therefore$ The necessary and sufficient conditions for the line to pass through the 1st quadrant is either $ac > 0$ or, $bc > 0$.

Q.92. Solution:-

$$\left.\begin{aligned} 2x+3y&=4 \\ ax-by&=7 \end{aligned}\right\} \text{———— (i)}$$

$\therefore \Delta_1 = \begin{vmatrix} 2 & 3 \\ a & -b \end{vmatrix} \neq 0$, since it has only one solution.

$\Rightarrow -(3a+2b) \neq 0 \Rightarrow (3a+2b) \neq 0.$

$$\left.\begin{aligned} 12x-8y&=7 \\ bx+ay&=0 \end{aligned}\right\} \text{———— (ii)}$$

$\therefore \Delta_2 = \begin{vmatrix} 12 & -8 \\ b & a \end{vmatrix} = 4(3a+2b)$, since $(3a+2b) \neq 0$,

so, $\Delta_2 \neq 0$; so, the equations in (ii) has only one solution.

Q.93. Solution:-

$$\text{LHS} = \frac{x_1}{x_5} + \frac{x_2}{x_5}$$

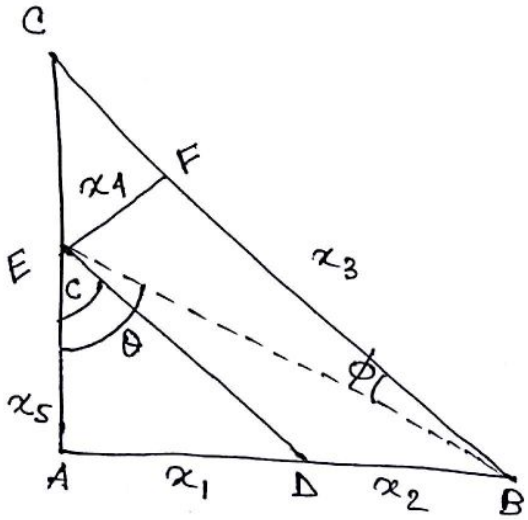
$$= \frac{x_1+x_2}{x_5}$$

$$= \tan\theta = \tan(\phi + c)$$

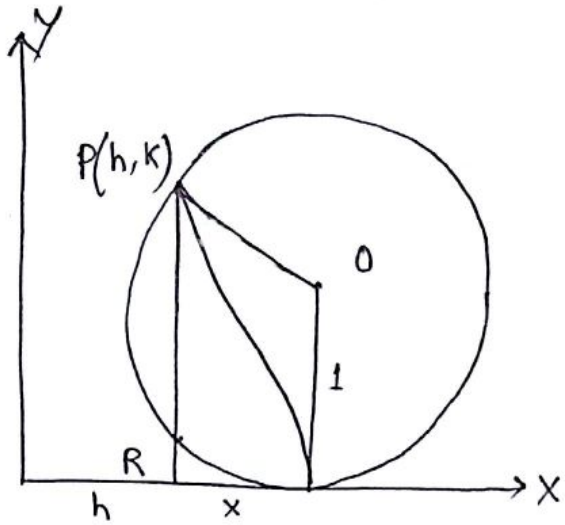
$$= \frac{\tan c + \tan\phi}{1 - \tan c \tan\phi}$$

$$= \frac{\frac{x_1}{x_5} + \frac{x_4}{x_3}}{1 - \frac{x_1}{x_5} \cdot \frac{x_4}{x_3}}$$

$$= \frac{x_1 x_3 + x_4 x_5}{x_3 x_5 - x_1 x_4}$$



Q.94.



Let at some point of time the point $P(0,0)$ which was initially at the circumference of the circle moved to point $P(h,K)$.

Let at that point O is the centre of the circle. Draw perpendicular to O to x-axis and P to X-axis and let they meet at Q and R respectively.

Now, $OP^2 = 1$.

$\Rightarrow (h+x-h)^2 + (K-1)^2 = 1$ since co-ordinate of $O = (h+x, 1)$

$\Rightarrow x^2 + K^2 = 2K$.

From triangle OPQ, we get, $PQ^2 = 1^2 + 1^2 - 2\cos(\angle POQ) = 2(1-\cos\theta)$

[let $\angle POQ = \theta$]

$\Rightarrow x^2 + K^2 = 2(1-\cos\theta)$

$\Rightarrow 2K = 2(1-\cos\theta)$

$\Rightarrow \theta = \cos^{-1}(1-K)$

Also, arc $PQ = x + h = 1 \times \theta = \theta$

$\Rightarrow x = \theta - h$.

From ①, we get, $x = \pm\sqrt{2K - K^2}$.

$\Rightarrow \theta - h = \pm\sqrt{2K - K^2}$.

$\Rightarrow \cos^{-1}(1-K) - h = \pm\sqrt{2K - K^2}$

$\Rightarrow \{\cos^{-1}(1-K) - h\}^2 = 2K - K^2$

$\therefore$ locus is, $\{\cos^{-1}(1-y) - x\}^2 = 2y - y^2$.

Q.95. Solution:-

Any circle passing through P and Q is $(x^2+y^2-2cy-a^2)+K(x^2+y^2-2bx+a^2)=0$

Now, this passes through $O=(0,0)$.

$\Rightarrow \quad -a^2+Ka^2=0$

$\Rightarrow \quad K=1.$

So, the equation of the circle passing through P, Q and O is

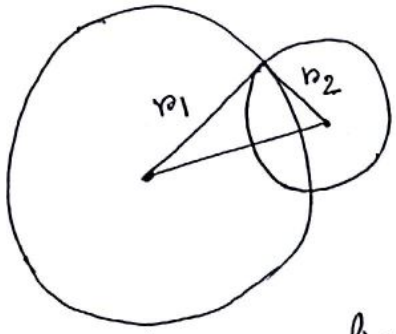
$$x^2+y^2-bx-cy=0.$$

Clearly, $A=(0,c)$ and $B=(b,0)$

Clearly, A and B satisfies the above equation of the circle.

$\Rightarrow$ P, Q, A, B and O lie on a circle.

Q.96. Solution:-



Let the equation of any circle passing through $(2,0)$ and $(-2,0)$ is $x^2+y^2+2gx+2fy+c=0$

Centre $=(-g,-f)$ and radius $=\sqrt{g^2+f^2-c}$

The given circle is, $\left(x-\frac{5}{2}\right)^2+y^2=\left(\frac{3}{2}\right)^2$

Now, from the figure, distance between centre is $=g^2+f^2-c+\left(\frac{3}{2}\right)^2$.

$$\Rightarrow \left(g-\frac{5}{2}\right)^2+f^2=g^2+f^2-c+\frac{9}{4}$$

$\Rightarrow \quad 5g-c=4.$

Now, $(2,0)$ and $(-2,0)$ satisfy the equation of the circle.

$\Rightarrow \quad 4+4g+c=0$ and $4-4g+c=0.$

Solving we get, $g=0$ and $c=-4$ which satisfy $5g-c=4.$

Q.97. (Solution)

The point on the circumference of the circle, the point on the parabola and the centre of the circle will be collinear in case of minimum distance between the circle and the parabola.

So, we will calculate minimum distance of the centre of the circle from the parabola and then we will subtract the radius to get the required result.

Any point on the parabola $= (t^2, 2t)$ and the centre of the circle $= (2, 8)$.

$$D^2 = (t^2-2)^2 + (2t-8)^2$$

$$\frac{dD^2}{dt} = 2(t^2-2)\times 2t + 2(2t-8)\times 2 = 0$$

$$\Rightarrow t^3 - 2t + 2t - 8 = 0$$

$$\Rightarrow t^3 = 8$$

$$\Rightarrow t = 2.$$

Now, $\frac{d^2D^2}{dt^2} = 3t^2 > 0$ at $t = 2$.

So, D^2 has minimum value of $t = 2$.

Therefore, $D^2_{min} = (2^2-2)^2 + (2\times 2 - 8)^2$

$= 20$

$$\Rightarrow D_{min} = 2\sqrt{5}.$$

Therefore required distance $= (2\sqrt{5} - 1)$ [$\because$ radius $= 1$]

Q. 102.

Let radius of the circle $= a$,

The co-ordinate axes are so chosen that the centre of the circle is the origin $= (0,0)$.

Let co-ordinate of point $A = (a\cos A, a\sin A)$ and $B = (a\cos(A+\theta), \sin(A+\theta))$

and that of P is $(a\cos\phi, a\sin\phi)$.

$$|AP| = a\sqrt{(\cos A - \cos\phi)^2 + (\sin A - \sin\phi)^2}$$

$$= a\sqrt{\{2 - \cos(A-\phi)\}}.$$

$$|BP| = a\sqrt{\{\cos(A+\theta) - \cos\phi\}^2 + \{\sin(A+\theta) - \sin\phi\}^2}$$

$$= a\sqrt{\{2 - \cos(A+\theta-\phi)\}}$$

Let $D_1 = |AP| \times |BP|$

$$= a^2\sqrt{\{2-\cos(A-\phi)\}\{2-\cos(A+\theta-\phi)\}}$$

Let $D = \dfrac{D_1^2}{a^2}$

D_1 will be maximum when D is maximum.

So, $$\frac{dD}{d\phi} = -\sin(A-\phi)\{2-\cos(A+\theta-\phi)\} - 2\sin(A+\theta-\phi)\{2-\cos(A-\phi)\}$$

$$= 0.$$

$$\Rightarrow \sin\left\{\frac{2A+\theta-2\phi}{2}\right\} = 0$$

$\Rightarrow \phi = A + \dfrac{\theta}{2}$; i.e., middle point of the arc AB.

Q.103. Solution:- $P(3,4)$; $Q(x, \sqrt{25-x^2})$

Slope, $M(x) = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{\sqrt{25-x^2} - 4}{x-3}$

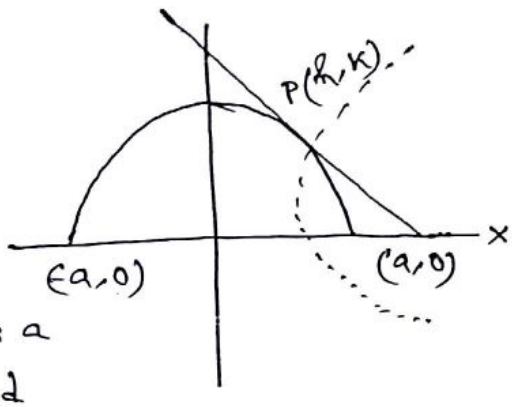
Now, $\lim\limits_{x \to 3} M(x)$

$= \lim\limits_{x \to 3} \dfrac{\sqrt{25-x^2} - 4}{x-3}$

$= -\dfrac{3}{4}$ [Do yourself applying L'Hospital Rule]

Q.104. Solution:-

(a) $AB = 2a = 3cm$, let, $P(h,k)$ be the coordinate of P and $PA = 2PB$



$\therefore (h+a)^2 + k^2 = 4\{(h-a)^2 + k^2\}$

$\Rightarrow 3h^2 + 3k^2 - 10ha + 3a^2 = 0$

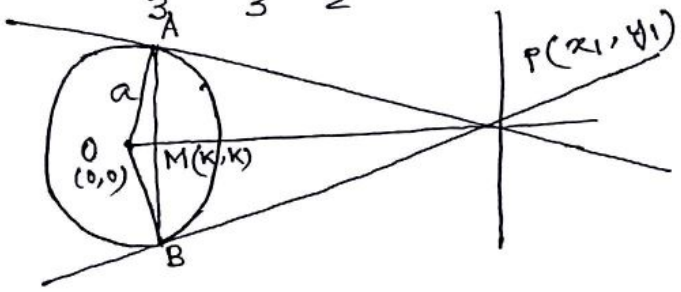
Locus of P is $x^2 + y^2 - \frac{10}{3}ax + a^2 = 0$

$\Rightarrow \left(x - \frac{5}{3}a\right)^2 + y^2 = \left(\frac{4a}{3}\right)^2$ which is a circle with centre at $\left(\frac{5}{3}a, 0\right)$ and radius $= \frac{4a}{3}$.

$\therefore$ Co-ordinates of K are $\left(\frac{5a}{3}, 0\right)$ & $KP = \frac{4a}{3}$.

$\therefore KB = OK - OB = \frac{5a}{3} - a = \frac{2a}{3} = \frac{2}{3} \times \frac{3}{2} = 1cm.$

Q.105. Solution:-



Slope of $OM = k/h$

[taking centre $(0,0)$]

$\therefore$ Slope of $AB = -h/k$.

[$\because AB \perp OM$]

$\therefore$ Equation of AB, whose slope is $-h/k$ and which passes through the point (h,k) is $y - k = -\frac{h}{k}(x-h)$

or, $hx + ky = h^2 + k^2$ .....(1)

And equation of AP, the tangent is $xx_1 + yy_1 = a^2$.....(2)

$\therefore$ From (1) and (2), we have,

$$\frac{x_1}{h} = \frac{y_1}{k} = \frac{a^2}{h^2+k^2}$$

$$\therefore x_1 = \frac{ha^2}{h^2+k^2}, \quad y_1 = \frac{ka^2}{h^2+k^2}.$$

$\therefore$ Put these values of x_1 and y_1 in $lx_1 + my_1 + n = 0$

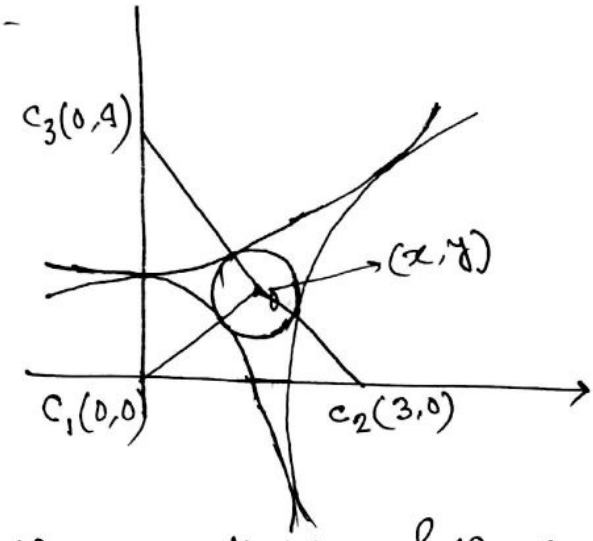
we get, $$l \cdot \frac{ha^2}{h^2+k^2} + m \cdot \frac{ka^2}{h^2+k^2} + n = 0$$

$$\Rightarrow lha^2 + mka^2 + n(h^2+k^2) = 0$$

$$\Rightarrow h^2 + k^2 + \frac{la^2}{n}h + \frac{ma^2}{n}k = 0$$, i.e., the required focus of M.

So, the equation of the circle is $x^2 + y^2 + \frac{la^2}{n}x + \frac{ma^2}{n}y = 0$

Q. 106. Solution:-



For simplicity,
centre of $C_1 \equiv (0,0)$
$\therefore C_2 \equiv (3,0)$
$\therefore C_3 \equiv (0,4)$

Let (x,y) be the co-ordinates of the centre of the circle touching C_1, C_2 and C_3 and let r be its radius.

Then,
$$(r+1)^2 = x^2 + y^2 \quad \ldots\ldots (1)$$
$$(r+2)^2 = (x-3)^2 + y^2 \quad \ldots\ldots (2)$$
$$(r+3)^2 = x^2 + (y-4)^2 \quad \ldots\ldots (3)$$

Solving these three equations, we will get the following equation:- $23x^2 - 90x + 63 = 0$

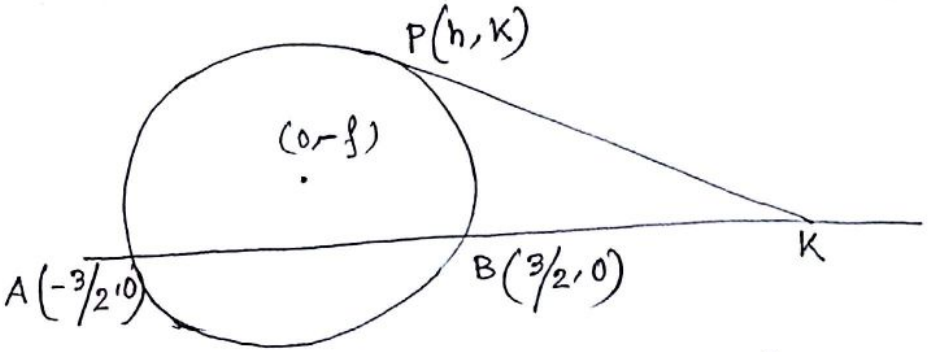
$$\therefore x = \frac{90 \pm \sqrt{90^2 - 4.23.63}}{2.23} = \frac{21}{23}, \text{ or } 3.$$

By the diagram, x can't be 3, so, $x = \frac{21}{23}$, $\therefore y = \frac{20}{23}$.

$\therefore$ Required centre of the circle is $\left(\frac{21}{23}, \frac{20}{23}\right)$.

Q.104.

(a)



Let the equation of the circle passing through A, B and P is

$x^2+y^2+2gx+2fy+c=0.$

Now, this circle passes through $A(-3/2, 0)$ and $B(3/2, 0)$

So, $(-3/2)^2-3g+c=0$ and $(3/2)^2+3g+c=0.$

Solving for g and c we get, $g=0$ and $c=-\frac{9}{4}.$

Putting back these values in the circle we get,

$x^2+y^2+2fy-9/4=0.$

Now, this circle passes through $P(h,k)$

So, we have, $h^2+k^2+2fk-9/4=0$ ——— (1)

Also, we have $PA=2PB$

$\Rightarrow PA^2=4PB^2$

$\Rightarrow (h+3/2)^2+k^2=4\{(h-3/2)^2+k^2\}$

$\Rightarrow h^2+k^2-5h+9/4=0$ ——— (2)

Now, slope of the line joining centre and P is $(k+f)/h$.

So, slope of PK is $-\frac{h}{(k+f)}$.

Equation of KP is, $y - k = \left\{\frac{-h}{(k+f)}\right\}(x-h)$

$\Rightarrow xh + (k+f)y = h^2 + k^2 + kf$

Putting $y = 0$ we get,

abscissa of K, $x = \frac{h^2+k^2+kf}{h}$.

So, coordinate of K is $\left\{\frac{h^2+k^2+kf}{h}, 0\right\}$

Now, $\frac{h^2+k^2+kf}{h} = \left[h^2+k^2+\frac{1}{2}\left\{\frac{9}{4}-(h^2+k^2)\right\}\right]/h$

[From ①]

$= \left\{\frac{1}{2}(h^2+k^2)+\frac{9}{8}\right\}/h$

$= \left\{\frac{1}{2}\left(5h - 9/4\right)+\frac{9}{8}\right\}/h$ [from ②]

$= \frac{5h}{2h} = \frac{5}{2}$.

So, coordinate of K is $(5/2, 0)$.

$KB = 5/2 - 3/2 = 1$.

$KP^2 = (5/2 - h)^2 + k^2$

$= 25/4 - 5h + h^2 + k^2$.

$= 25/4 - 9/4$ [from ②] $= 4 \Rightarrow KP = 2$

Q.104. (b) Solution

Let $A(0,0)$ and $B(3,0)$ and $P(h,k)$

Now, $PA = 2PB$

$\Rightarrow (h^2+k^2) = 4\{(h-3)^2+k^2\}$

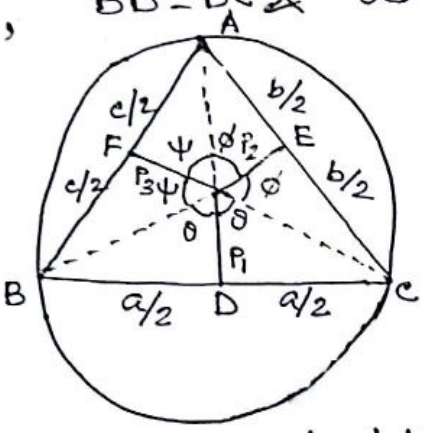
$\Rightarrow h^2+k^2-8h+12 = 0$

Putting $h = x$ and $y = k$, we get the required locus

as, $x^2+y^2-8x+12 = 0$ which is a circle.

Q.107. Solution:- As P_1, P_2, P_3 are the lengths of the perpendiculars drawn from the circumcentre O to the sides of length a, b, c respectively, then from the diagram. D, E, F are the mid points of BC, CA, AB respectively. Hence, in ΔBOD and ΔCOD,

$\angle BDO = \angle CDO$, $BD = DC$ & OD is common. $\therefore \Delta BOD \equiv \Delta COD$.



Similarly, $\Delta COE \equiv \Delta AOE$ and $\Delta AOF \equiv \Delta BOF$,

$\Rightarrow$ $\angle BOD = \angle COD = \theta$, say
$\angle COE = \angle AOE = \phi$, say
$\angle AOF = \angle BOF = \psi$, say.

$\therefore$ $\angle BOD + \angle COD + \angle COE + \angle AOE + \angle AOF + \angle BOF$
$= 2(\theta + \phi + \psi) = 2\pi$

$\therefore$ $\theta + \phi = \pi - \psi$

$\Rightarrow$ $\tan(\theta + \phi) = \tan(\pi - \psi) = -\tan\psi$.

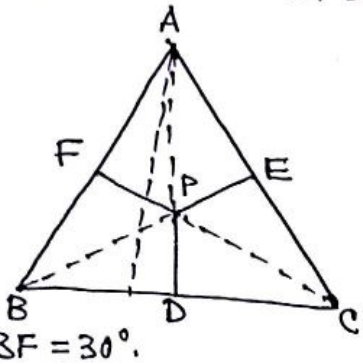
Hence we can show, $\tan\theta + \tan\phi + \tan\psi = \tan\theta\tan\psi\tan\phi$

i.e. $$\frac{a}{2p_1} + \frac{b}{2p_2} + \frac{c}{2p_3} = \frac{abc}{8p_1p_2p_3}$$

or, $$\frac{a}{p_1} + \frac{b}{p_2} + \frac{c}{p_3} = \frac{abc}{4p_1p_2p_3}.$$

Q.108. Solution:-

We have (i) $AB = BC = CA$ &
$\angle B = \angle C = \angle A = 60°$.
[$\because \Delta ABC$ is equilateral]



(ii) $PD \perp BC$, $PE \perp AC$, $PF \perp AB$.
So, from (i) $\angle PAF = \angle PAE = 30°$,
$\angle PCE = \angle PCD = 30°$, $\angle PBD = \angle PBF = 30°$.

$\therefore$ In ΔPAF, $\dfrac{PF}{AF} = \tan 30^\circ = \dfrac{1}{\sqrt{3}}$

or, $PF = \dfrac{AF}{\sqrt{3}}$.

Similarly, for ΔPBD, we get $PD = \dfrac{1}{\sqrt{3}} BD$, and for ΔPCE, we get $PE = \dfrac{1}{\sqrt{3}} CE$.

$$\therefore \frac{PD + PE + PF}{BD + CE + AF} = \frac{\frac{1}{\sqrt{3}}(BD + CE + AF)}{BD + CE + AF} = \frac{1}{\sqrt{3}}.$$

As each of the PD, PE, PF can be represented w.r.t. BD, CE, AF respectively, so the specified ratio does not depend upon the choice of the point P.

Q. 111. (a) Let $PA = x$ and $PB = y$

Solution:-

$$\therefore PA.PB = xy = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2$$

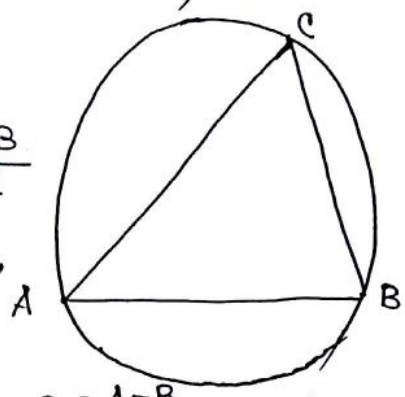
xy is maximum when $x = y$.

i.e., $PA = PB$ $\Rightarrow$ ΔPAB will be an isosceles one.

Position of P:- P will be the point of intersection of the perpendicular bisector of AB and the circle C.

(b) $$\frac{AP}{\sin B} = \frac{BP}{\sin A} = \frac{AB}{\sin P}$$

$$AP = \frac{AB}{\sin P} \times \sin B; \quad BP = \frac{AB}{\sin P} \times \sin A;$$



$$\therefore AP + BP = \frac{AB}{\sin P} \times (\sin A + \sin B)$$

$$= \frac{AB}{\sin P} \times 2 \sin\frac{A+B}{2} \cos\frac{A-B}{2}$$

$$= \frac{AB}{\sin P} . 2\cos\frac{P}{2} \cos\frac{A-B}{2}.$$

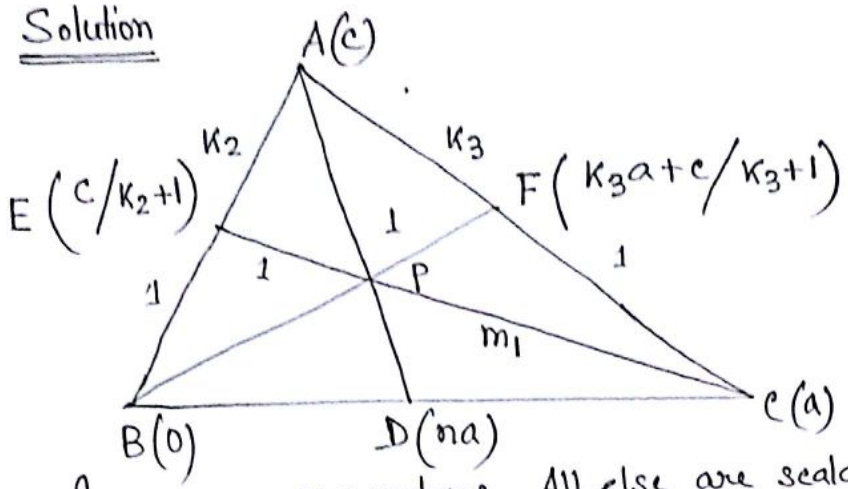
$\because$ AB = constant, $\angle P$ = constant,

$\therefore$ $AP + BP$ = maximum when $\cos\dfrac{A-B}{2}$ = maximum $= 1$.

$\Rightarrow \dfrac{A-B}{2} = 0 \Rightarrow A = B \Rightarrow PA = PB$.

$\Rightarrow$ P lies on the point of intersection of the bisector of AB.

Q.109. Solution



In the figure, a, c are vectors. All else are scalars.

Now, $P = \dfrac{m_1 c/(K_2+1) + a}{m_1+1} = \dfrac{m_2\left\{\dfrac{K_3a+c}{K_3+1}\right\}}{m_2+1}$.

Equating, the coefficients of vectors a and c from both sides we get,

$$\left\{\frac{m_1}{m_1+1}\right\}\Big/(K_2+1) = \left\{\frac{m_2}{m_2+1}\right\}\Big/(K_3+1) \quad \text{and}$$

$$\frac{1}{m_1+1} = \frac{m_2K_3}{(K_3+1)(m_2+1)}.$$

$$\Rightarrow \quad m_1 = \frac{K_2+1}{K_3}.$$

Therefore, $P = \dfrac{c+K_3a}{K_2+K_3+1}$.

Let $AP/PD = K_4$.

Then, $P = \dfrac{K_4na+c}{K_4+1}$.

$$\therefore \quad \frac{c+K_3a}{K_2+K_3+1} = \frac{K_4na+c}{K_4+1}.$$

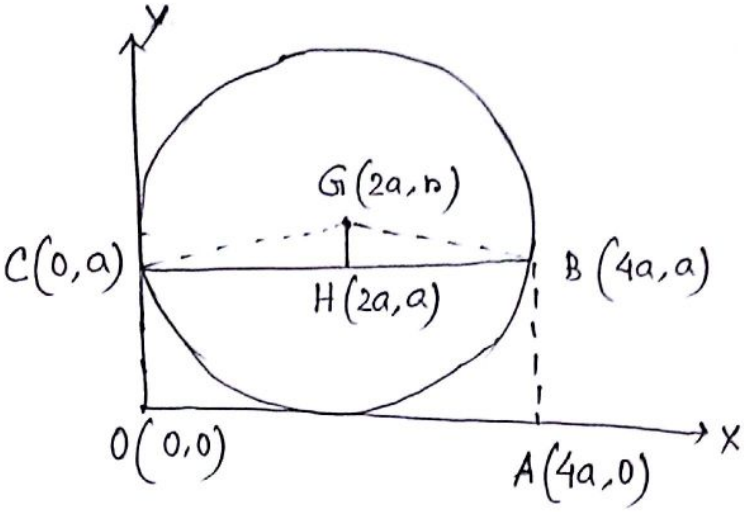
Equating coefficients of vector c from both sides we get,

$$\frac{1}{K_2+K_3+1} = \frac{1}{K_4+1}.$$

$$\Rightarrow \quad K_2+K_3 = K_4.$$

$$\Rightarrow \quad \frac{AF}{FB} + \frac{AE}{EC} = \frac{AP}{PD}.$$

Q.112. Solution



Now, $GC = \sqrt{(2a-0)^2 + (r-a)^2} = r$

$\Rightarrow r = \frac{5a}{2}$.

Let $\angle CGB = \theta$

From ΔCGB, we get $\frac{5a}{2\sin(90^\circ - \theta/2)} = \frac{4a}{\sin\theta}$

$\Rightarrow \sin\frac{\theta}{2} = \frac{4}{5} \Rightarrow \theta = 2\sin^{-1}\left(\frac{4}{5}\right)$

$GH = \frac{5a}{2} - a = \frac{3a}{2}$.

$\text{Area}(\Delta GHC) = \frac{1}{2} \times \frac{3a}{2} \times 4a = 3a^2$.

Area of joining curve G, C, mid point of OA and B is

$\frac{1}{2\pi}\pi\left(\frac{5a}{2}\right)^2\theta = \frac{25a^2}{4}\sin^{-1}\left(\frac{4}{5}\right)$.

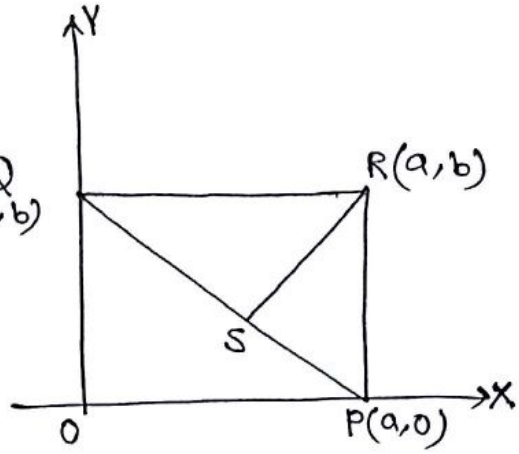
Area of the part BHC and mid point of joining OA is

$\frac{25a^2}{4}\sin^{-1}\left(\frac{4}{5}\right) - 3a^2$.

Area of each small part of the triangle (i.e., point joining O, C and mid-point of OA) $= \frac{1}{2}\left[4a^2 - \left\{\frac{25a^2}{4}\sin^{-1}\left(\frac{4}{5}\right) - 3a^2\right\}\right]$

Hence find the ratio.

Q.114. Solution:-



Equation of PQ: $\frac{x}{a}+\frac{y}{b}=1$.

$\therefore bx+ay-ab=0$ ·····①

As PQ = L, so $a^2+b^2=L^2$ ····②

Equation of the line through R(a,b) and perpendicular to PQ is

$y-b=\frac{a}{b}(x-a)$

or, $ax-by-(a^2-b^2)=0$ ·····③

Both the lines PQ & RS meet at point S, whose locus we are to find, the variables being a, b, which are connected by $a^2+b^2=L^2$.

Solving ① and ③, we have

$$\frac{x}{-a^3+ab^2-ab^2}=\frac{y}{-a^2b+a^2b-b^3}=\frac{1}{-b^2-a^2}$$

$$\Rightarrow \frac{x}{-a^3}=\frac{y}{-b^3}=-\frac{1}{b^2+a^2}=-\frac{1}{L^2};$$

$$\Rightarrow a=(L^2x)^{1/3} \;\&\; b=(L^2y)^{1/3};$$

$\because a^2+b^2=L^2$

$$\therefore (L^2x)^{2/3}+(L^2y)^{2/3}=(L^2)^{2/3}$$

$\therefore x^{2/3}+y^{2/3}=L^{2/3}$ which is the required locus of S.

Q.115. Solution:-

$$\frac{\sin^4 x}{a}+\frac{\cos^4 x}{b}=\frac{1}{a+b},$$

$$\text{or, } \frac{\sin^4 x}{a}+\frac{(1-\sin^2 x)^2}{b}=\frac{1}{a+b},$$

$$\text{or, } (a+b)^2\sin^4 x-2a\sin^2 x(a+b)+a^2=0$$

$$\therefore \{(a+b)\sin^2 x-a\}^2=0$$

$\therefore \sin^2 x=\frac{a}{a+b}$ $\quad\therefore \cos^2 x=1-\frac{a}{a+b}=\frac{b}{a+b};$

$$\therefore \frac{\sin^6 x}{a^2}+\frac{\cos^6 x}{b^2}=\frac{\frac{a^3}{(a+b)^3}}{a^2}+\frac{\frac{b^3}{(a+b)^3}}{b^2}=\frac{1}{(a+b)^2}.$$

Q.116. Solution:- $\sin A + \sin B - \sin C$

$= \sin A + \sin B - \sin\left(\frac{\pi}{2} - C\right)$

$= \sin A + \sin B + \sin\left(C - \frac{\pi}{2}\right)$

$= \sin A + \sin B + \sin D, \quad D = C - \frac{\pi}{2}.$

Now, $A+B+C=\pi$; $\therefore A+B+C-\frac{\pi}{2} = A+B+D = \frac{\pi}{2}$.

Let, $f(x) = \sin x$, we plot its graph such that taking the abcissa A, B, D as $A+B+D = \frac{\pi}{2}$, or, plotting in the interval $[0, \frac{\pi}{2}]$

$\therefore$ Centroid of $\Delta PQR \equiv G \equiv \left(\frac{A+B+D}{3}, \frac{\sin A + \sin B + \sin D}{3}\right)$.

We take a point $f(x) = \sin x$, such that it is of the same abcissa that of G, but of greater ordinate.

$\therefore M \equiv \left(\frac{A+B+D}{3}, \sin\frac{A+B+C}{3}\right)$.

$\therefore$ Ordinate of G < ordinate of M,

$\Rightarrow \frac{\sin A + \sin B + \sin D}{3} < \sin\frac{A+B+D}{3}$,

or, $\sin A + \sin B + \sin D < \frac{3}{2}$.

When A, B, D are not distinct, i.e., $A = B = D = \frac{\pi}{6}$, equality holds. $\Rightarrow \sin A + \sin B + \sin D \le \frac{3}{2}$, $\Rightarrow \sin A + \sin B - \cos C \le \frac{3}{2}$.

Q.118. Solution:- Let, a = radius of the circle,

$AX = x = BC$. Since, $AB.BX = AX^2$,

$\Rightarrow a(a-x) = x^2 \Rightarrow a^2 - ax - x^2 = 0$,

$\Rightarrow \left(\frac{x}{a}\right)^2 + \frac{x}{a} - 1 = 0$,

$\therefore \frac{x}{a} = \frac{-1+\sqrt{5}}{2}$.

Now, $x = BC = 2a\sin\frac{\theta}{2}$ $\therefore \sin\frac{\theta}{2} = \frac{x}{2a} = \frac{-1+\sqrt{5}}{4} = \sin 18^\circ$

$\Rightarrow \frac{\theta}{2} = 18^\circ \Rightarrow \theta = 36^\circ$.

$\Rightarrow \angle BAC = 36^\circ$.

Q.119. Solution

Let $\angle AZC = \theta$.

Now, angle $ZAC = 90^\circ - \alpha$

$\Rightarrow \angle CAZ = 180^\circ - (90^\circ - \alpha) - \theta = 90^\circ - \alpha - \theta$

From ΔAZC, we get, $\dfrac{CZ}{\sin(90^\circ + \alpha - \theta)} = \dfrac{AZ}{\sin(90^\circ - \alpha)}$

$$\Rightarrow \frac{CZ}{\cos(\alpha - \theta)} = \frac{AZ}{\cos\alpha} \quad \text{------} (1)$$

$\angle ZCB = 90^\circ - \beta$.

$\Rightarrow \angle CBZ = 180^\circ - (180^\circ - \theta) - (90^\circ - \beta) = -\{90^\circ - (\theta + \beta)\}$.

From ΔBZC, we get, $\dfrac{CZ}{\sin[-\{90^\circ - (\theta+\beta)\}]} = \dfrac{BZ}{\sin(90^\circ - \beta)}$

$$\Rightarrow \frac{CZ}{\{-\cos(\theta+\beta)\}} = \frac{BZ}{\cos\beta} \quad \text{--------} (2)$$

Dividing (1) by (2), we get,

$$\frac{\{-\cos(\theta+\beta)\}}{\cos(\alpha-\theta)} = \frac{AZ\cos\beta}{BZ\cos\alpha}.$$

$$\Rightarrow \frac{\sin\theta\sin\beta - \cos\theta\cos\beta}{\cos\alpha\cos\theta + \sin\alpha\sin\theta} = \frac{\cos\beta}{2\cos\alpha} \quad [\because AZ/BZ = 1/2]$$

$$\Rightarrow \frac{\tan\theta\sin\beta - \cos\beta}{\cos\alpha + \sin\alpha\tan\theta} = \frac{\cos\beta}{2\cos\alpha}$$

(After some computation)

$\Rightarrow$ .....

$\Rightarrow$ .....

$$\Rightarrow \tan\theta = \frac{3\cos\alpha\cos\beta}{2\cos\alpha\sin\beta - \sin\alpha\cos\beta}.$$

Q.117. Solution:- $y = f(x) = [x] + \sqrt{x - [x]} = [x] + \sqrt{\{x\}}$

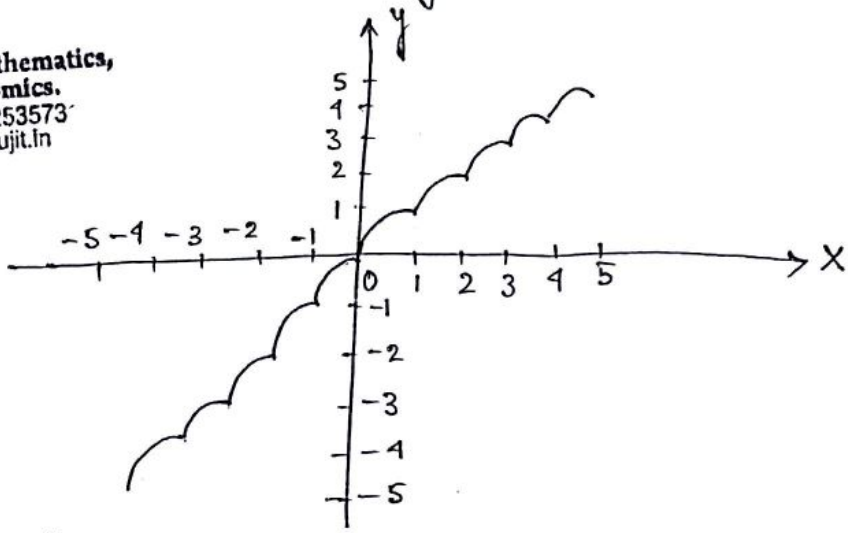
(i) We know $0 \le \{x\} < 1$

$\Rightarrow \sqrt{\{x\}} \geq \{x\}$

$\Rightarrow [x] + \sqrt{\{x\}} \geq [x] + \{x\}$

$\Rightarrow f(x) \geq x$

'=' holds when x takes integral values.



(ii) Again, $y = f(x) = [x] + \sqrt{\{x\}}$

As, $0 \le \{x\} < 1$, hence $\sqrt{\{x\}}$ is always real,

$\Rightarrow f(x)$ is always real.

$\Rightarrow$ there is a x_0, $\forall\ y_0 \in \mathbb{R}$ & $x_0 \in \mathbb{R}$, $\ni y_0 = f(x_0)$.

Q.121. Solution:- $x_1 = 2$, $x_2 = 1$, $2x_n - 3x_{n-1} + x_{n-2} = 0$.

Let, $x_n = ka^n$, $\therefore 2ka^n - 3ka^{n-1} + ka^{n-2} = 0$.

or, $2a^2 - 3a + 1 = 0$

or, $(2a-1)(a-1) = 0$

or, $a_1 = \frac{1}{2}, a_2 = 1$.

$\therefore x_n = K_1 a_1^n + K_2 a_2^n = K_1 \left(\frac{1}{2}\right)^n + K_2 (1)^n$.

Again, $x_1 = 2 = K_1 \left(\frac{1}{2}\right) \neq K_2 (1)^1 = \frac{K_1}{2} + K_2$ ..... (1)

$x_2 = 1 = K_1 \left(\frac{1}{2}\right)^2 + K_2 (1)^2 = \frac{K_1}{4} + K_2$ ..... (2)

Solving (1) and (2), we get $K_1 = 4$, $K_2 = 0$.

$\therefore x_n = 4 \left(\frac{1}{2}\right)^n = \frac{1}{2^{n-2}}$.

Q. 120.

(i) Now, $\sin 2A + \sin 2B + \sin 2C$

$$= 2\sin(A+B)\cos(A-B) + \sin\{2(A+B)\}$$

[As $A+B+C = n\pi$]

$$= 2\sin(A+B)\cos(A-B) + 2\sin(A+B)\cos(A+B)$$

$$= 2\sin(A+B)\{\cos(A-B) + \cos(A+B)\}$$

$$= 2(-1)^{n-1}\sin C \times 2\sin A \sin B \quad (\text{As } A+B+C = n\pi)$$

$$= (-1)^{n-1} 4\sin A \sin B \sin C$$

(ii) Let radius of the circum-circle is R.

Perimeter is same.

$\Rightarrow a+b+c = d+e+f$ (usual notation)

$$\Rightarrow \frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R} = \frac{d}{2R} + \frac{e}{2R} + \frac{f}{2R}$$

$$\Rightarrow \sin A + \sin B + \sin C = \sin D + \sin E + \sin F.$$

Q.121. Solution:-

$$2x_n - 3x_{n-1} + x_{n-2} = 0$$

$$\Rightarrow 2x_n - 2x_{n-1} = x_{n-1} - x_{n-2}.$$

Putting $n = n-1$, we get, $2x_{n-1} - 2x_{n-2} = x_{n-2} - x_{n-3}$

Putting $n = n-2$, we get, $2x_{n-2} - 2x_{n-3} = x_{n-3} - x_{n-4}$

. . . . . .

. . . . . .

Putting $n = 3$, we get, $2x_3 - 2x_2 = x_2 - x_1$

Adding add these together, $2x_n - 2x_2 = x_{n-1} - x_1$

$$\Rightarrow 2x_n - x_{n-1} = 2x_2 - x_1 = 0$$

$$\Rightarrow 2x_n = x_{n-1}$$

$$\Rightarrow x_n = \left(\frac{1}{2}\right)x_{n-1} = \left(\frac{1}{2}\right)^2 x_{n-2}$$

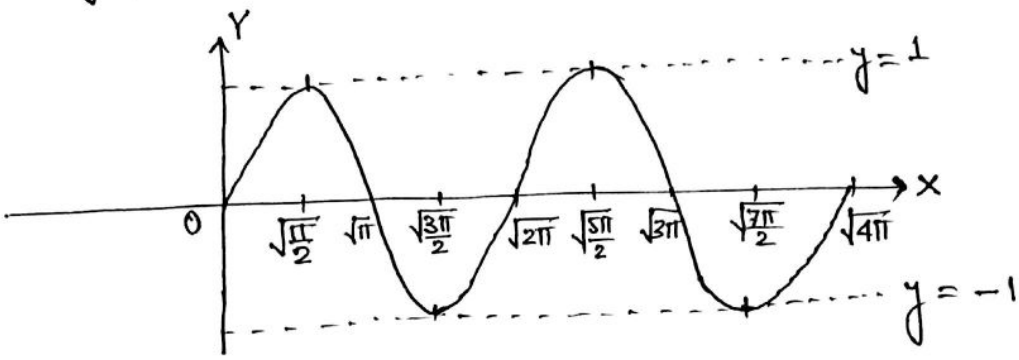
$$= \left(\frac{1}{2}\right)^3 x_{n-3}$$

. . . . .

$$= \left(\frac{1}{2}\right)^{n-2}.$$

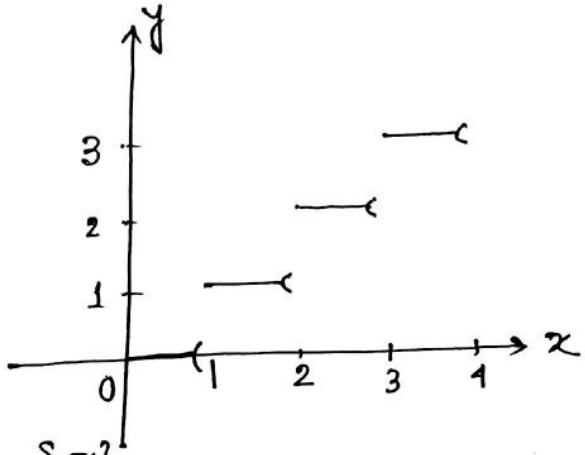
Q.122. **Solution:-** Taking different values of x, we get different y.

x	0	$\sqrt{\pi/4}$	$\sqrt{\pi/2}$	$\sqrt{\pi}$	$\sqrt{\frac{3\pi}{2}}$	$\sqrt{2\pi}$	$\sqrt{\frac{5\pi}{2}}$	$\sqrt{3\pi}$	$\sqrt{\frac{7\pi}{2}}$	$\sqrt{4\pi}$
y	0	$1/\sqrt{2}$	1	0	-1	0	1	0	-1	0



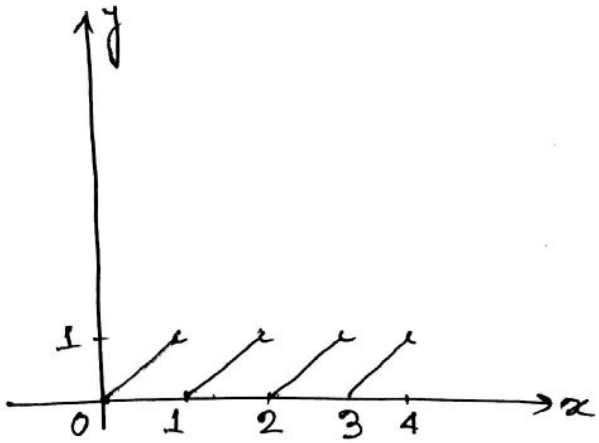
Q.123. **Solution:-** (i) $y = f(x) = [x]$

x	y
$[0,1)$	0
$[1,2)$	1
$[2,3)$	2
$[3,4)$	3
$\vdots$	$\vdots$



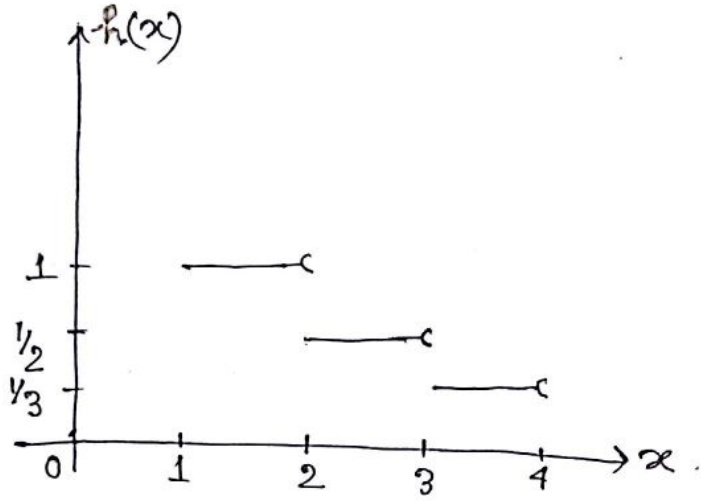
(ii) $g(x) = x - [x] = \{x\}$.

x	$g(x)$
0	0
$1/4$	$1/4$
$1/2$	$1/2$
$3/4$	$3/4$
1	0
$1\frac{1}{4}$	$1/4$
$1\frac{1}{2}$	$\frac{1}{2}$
$\vdots$	$\vdots$



(iii) $h(x) = \frac{1}{[x]}$

x	$h(x)$
1	1
$1\frac{1}{4}$	1
$1\frac{1}{2}$	1
$1\frac{3}{4}$	1
2	$\frac{1}{2}$
$2\frac{1}{2}$	$\frac{1}{2}$
$2\frac{3}{4}$	$\frac{1}{2}$
3	$\frac{1}{3}$



Q.124. Solution:- $y = \frac{x^2+1}{x^2-1} = \frac{(x^2-1)+2}{x^2-1}$

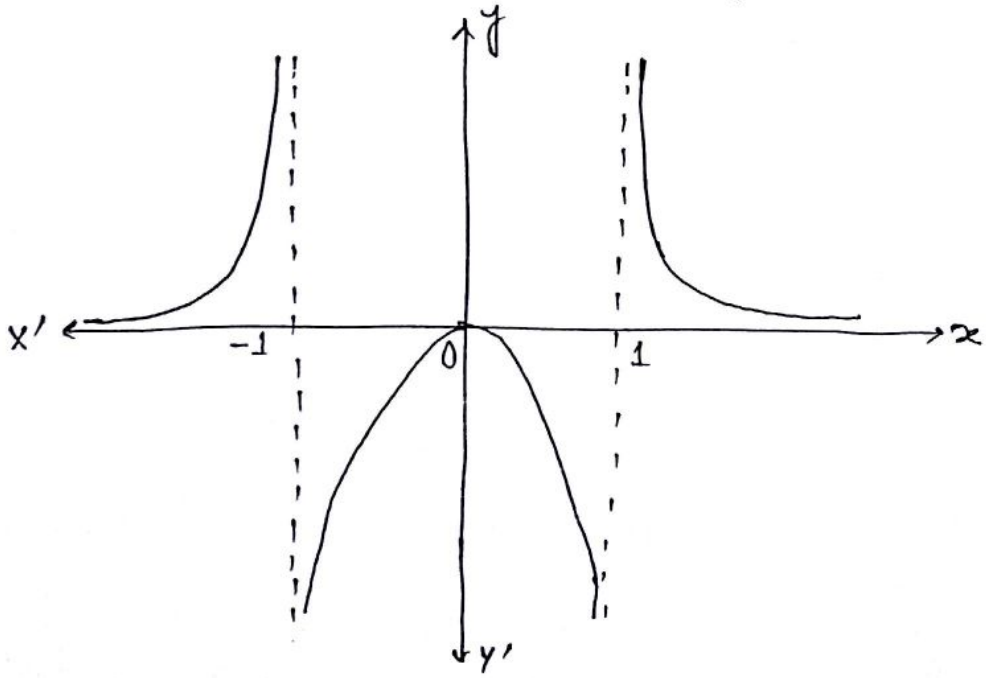
$\therefore y = 1 + \frac{2}{x^2-1}$ $\therefore \frac{dy}{dx} = -\frac{4x}{(x^2-1)^2}$

For, $-\infty \le x < -1$, y' is positive.

For, $-1 < x \le 0$, y' is positive

For, $0 \le x < 1$, y' is negative

For, $1 < x \le \infty$, y' is negative



Q.125. Solution

Now, $f(0)=1,\ f(1)=2,\ f(2)=1,\ f(3)=2,\ f(4)=3,\ f(5)=5,$

$f(6)=8,\ f(7)=13, \ldots\ldots$

Let, $f(n) > f(n+1)$.

Now, $f(n+1) = f(n) + f(n-1)$

$\Rightarrow f(n) - f(n+1) = -f(n-1)$

$\Rightarrow -f(n-1) > 0$

$\Rightarrow f(n-1) < 0$ which is possible for $n \geq 2$.

$\Rightarrow f(n) < f(n+1)$

Clearly, for $n = 0, 1, 2, 5$; $f(f(n)) = f(n)$.

Now, $f(n) = f(n-1) + f(n-2) = 2f(n-2) + f(n-3)$

$= 3f(n-3) + 2f(n-4)$

$= 5f(n-4) + 3f(n-5)$

$\Rightarrow f(n) \equiv 3f(n-5) \pmod 5$

$\Rightarrow f(5n) \equiv 3f(5(n-1)) \pmod 5$ which is true for $n=1$. and $f(0)$ and $f(5)$ both are divisible by 5.

$\Rightarrow f(5n) \equiv 0 \pmod 5$ for all n. (By Induction)

Q.126. Solution:- (i) $y = \sin x$, when $y > 0$ ------ $\boxed{\text{case-I}}$

$= 0$, when $y = 0$ ------ $\boxed{\text{Case-II}}$

$= -\sin x$, when $y < 0$ ------ $\boxed{\text{Case-III}}$

$\boxed{\text{Case-I}}$

x	$(4n+1)\frac{\pi}{2}$	$n\pi + (-1)^n \frac{\pi}{6}$
y	1	$1/2$

$\boxed{\text{Case-III}}$

x	$(4n+1)\frac{\pi}{2}$	$n\pi + (-1)^n \frac{\pi}{6}$
y	-1	$-1/2$

For all $n = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$

(ii) $|x| - |y| \geq 1$.

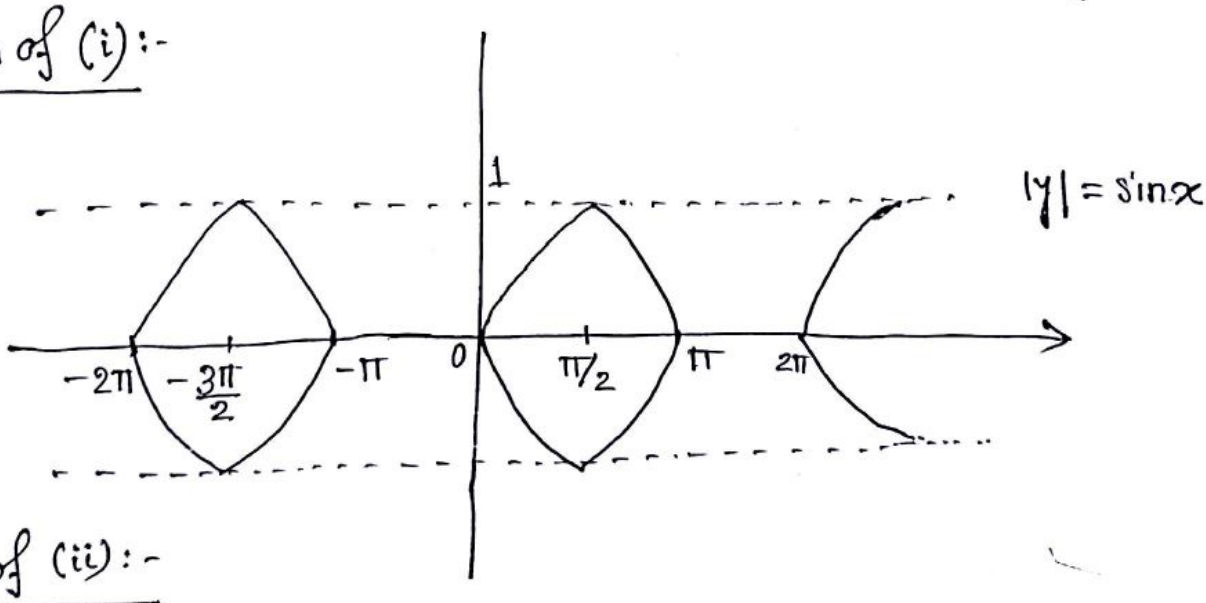
There are four cases :- (I) $x - y \geq 1$ when $x > 1, y > 0$

(II) $x + y \geq 1$ when $x \geq 1, y < 0$

(III) $-x - y \geq 1$ when $x < 0, y > 0$

(IV) $-x + y \geq 1$ when $x < 0, y < 0$

Graph of (i):-



Graph of (ii):-

Do yourself.

Q.128. Solution:-

$$y = |x| - 1$$

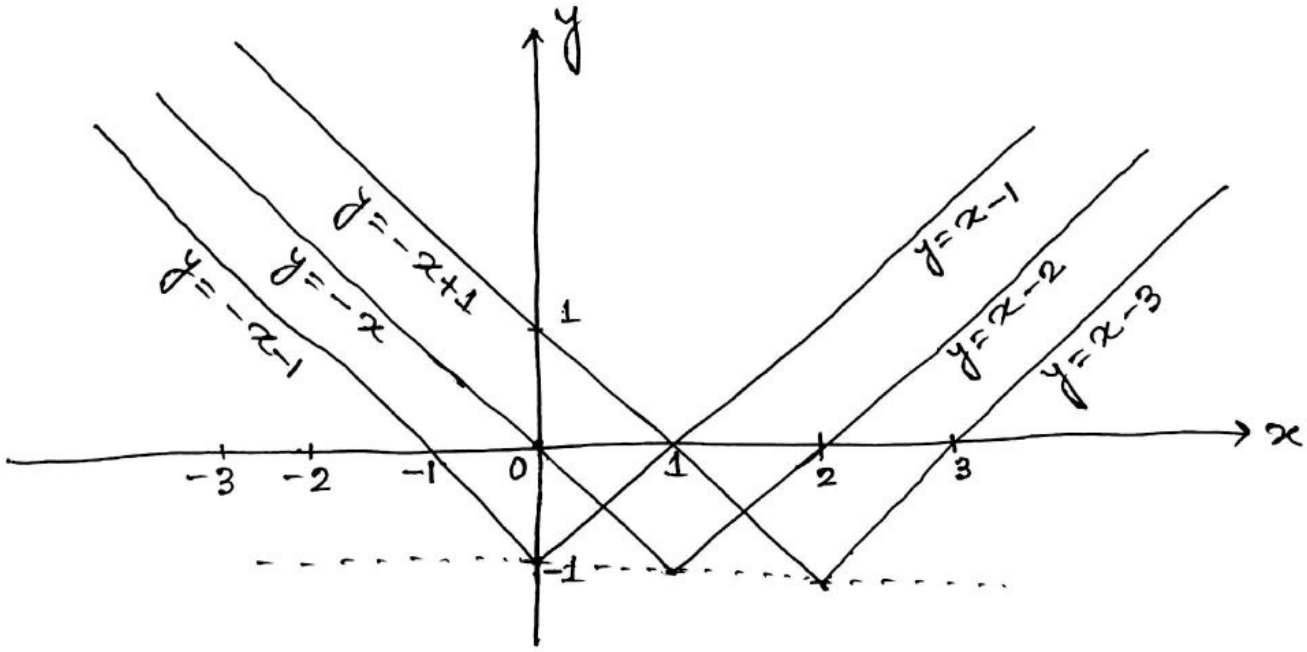
$$= \begin{cases} x-1, & \text{when } x \geq 0 \\ -x-1, & \text{when } x < 0 \end{cases}$$

$$Z = |x-1| - 1$$

$$= \begin{cases} x-1-1, & \text{when } x \geq 1 \\ -x+1-1, & \text{when } x < 1 \end{cases}$$

$$W = |x-2| - 1$$

$$= \begin{cases} x-2-1, & \text{when } x \geq 2 \\ -x+2-1, & \text{when } x < 2 \end{cases}$$



Q.129. Solution:-

$$f(x) = \frac{5-3x^2}{1-x^2}$$

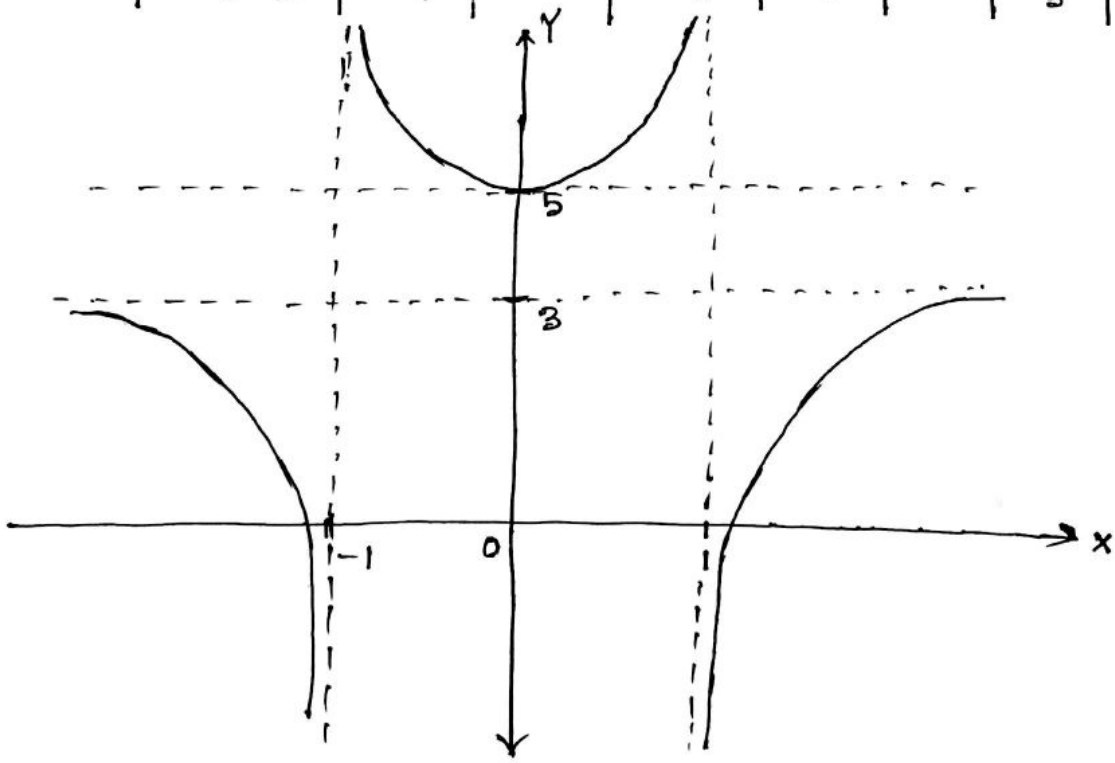
$$= \frac{3(1-x^2)+2}{1-x^2}$$

$$= 3 + \frac{2}{1-x^2}$$

$$\therefore f'(x) = \frac{4x}{(1-x^2)^2},$$

for $-\infty < x < -1$, $f'(x)$ is negative;
for $-1 < x \leq 0$, $f'(x)$ is negative;
for $0 < x < 1$, $f'(x)$ is positive;
for $1 < x < \infty$, $f'(x)$ is positive;

x	-3	3	-2	2	0	$-\frac{1}{2}$	$\frac{1}{2}$
$f(x)$	$2\frac{3}{4}$	$2\frac{3}{4}$	$2\frac{1}{3}$	$2\frac{1}{3}$	5	$\frac{17}{3}$	$\frac{17}{3}$

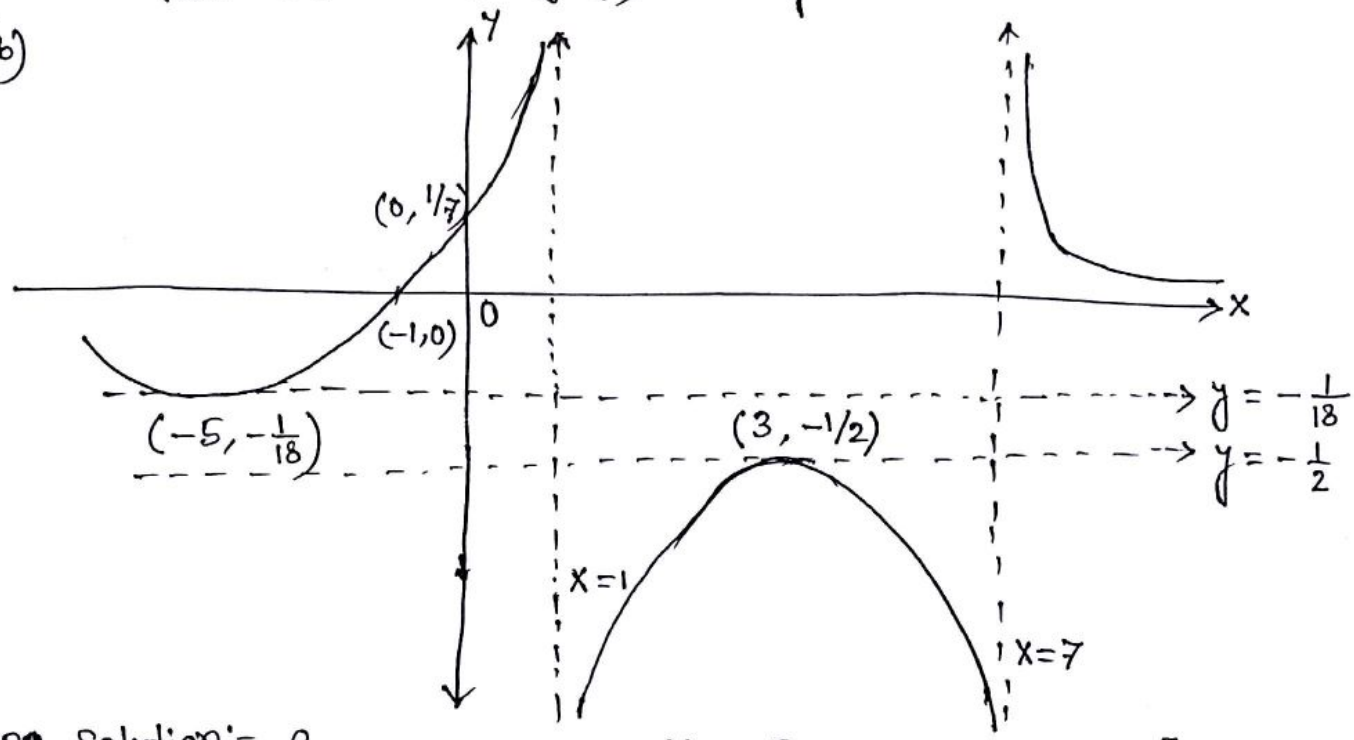


Q.130. (a) **Solution:-** $f(x) = \dfrac{x+1}{(x-1)(x-7)} = -\dfrac{1}{3}\cdot\dfrac{1}{x+1} + \dfrac{4}{3}\cdot\dfrac{1}{x-7}$.

$$f'(x) = \frac{1}{3}\cdot\frac{1}{(x+1)^2} - \frac{4}{3}\cdot\frac{1}{(x-7)^2}.$$

For $0 \le x < 1$, $f'(x)$ is positive,
For $-\infty < x < 0$, $f'(x)$ is negative,
For $1 < x < 7$, $f'(x)$ is negative,
For $7 < x < \infty$, $f'(x)$ is negative.

(b)



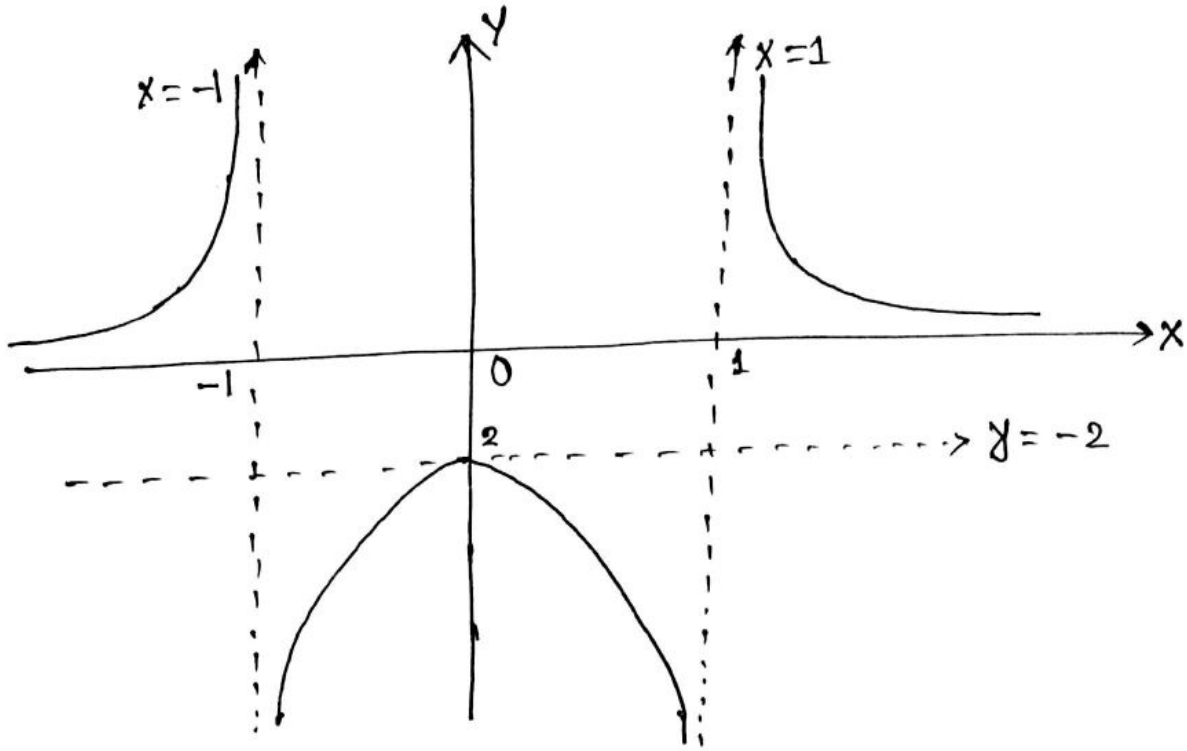
Q.132. **Solution:-** $f(x) = \dfrac{x-1}{x+1} + \dfrac{x+1}{x-1}$ [For $x \neq 1$, $x \neq -1$]

$$= \frac{2(x^2+1)}{(x^2-1)} = 2 + \frac{4}{x^2-1}.$$

$$\therefore f'(x) = \frac{-8x}{(x^2-1)^2}.$$

For, $-\infty < x < -1$, $f'(x)$ is positive,
For, $-1 < x < 0$, $f'(x)$ is positive,
For, $0 < x < 1$, $f'(x)$ is negative,
For, $1 < x < \infty$, $f'(x)$ is negative.

x	-3	-2	0	$-1/2$	$1/2$	2	3
$f(x)$	$2\frac{1}{2}$	$3\frac{1}{3}$	-2	$-3\frac{1}{3}$	$3\frac{1}{3}$	$3\frac{1}{3}$	2



Q.137. Solution:-

$$13[x] + 25\langle x\rangle = 271$$

$$\text{or, } \langle x\rangle = \frac{271 - 13[x]}{25},$$

We know $0 \le \langle x\rangle < 1$,

$$\Rightarrow 0 \le \frac{271 - 13[x]}{25} < 1$$

$$\Rightarrow 13[x] - 271 > -25$$

$$\Rightarrow [x] > 18.9$$

The nearest integers in this interval are 19 and 20.

Putting $[x] = 19$, $\langle x\rangle = \frac{271 - 13\times 19}{25} = 0.96$

Putting $[x] = 20$, $\langle x\rangle = \frac{271 - 13\times 20}{25} = 0.44$

$\therefore x = 19.96, 20.44$ (Answer)

Q.135. Solution:-

$$f(x) = \int_0^x f(t)\,dt$$

$$\Rightarrow f(0) = \int_0^0 f(t)\,dt.$$

$$\Rightarrow f(0) = 0.$$

$$\Rightarrow f(0) = k \times e^0 = 0 \quad (\text{from } (*))$$

$$\Rightarrow k = 0$$

$$\Rightarrow f(x) = 0 \ \forall\ x.$$

$$f(x) = \int_0^x f(t)\,dt$$

$$\Rightarrow f'(x) = f(x)$$

$$\Rightarrow \int \frac{f'(x)}{f(x)}\,dx = \int dx$$

$$\Rightarrow \ln|f(x)| = x + c$$

$$\Rightarrow f(x) = e^{x+c}.$$

$$\Rightarrow f(x) = ke^x \quad \text{-----} (*)$$

Q.145. (Solution)

Given $f''(x) > 0 \ \forall\ x$.

i.e. graph of $f(x)$ lies above any of it's tangent line.

Also, $f'(x) < 0$ for some x and $f'(0) > 0$ for some x.

Let $g(x) = f'(x)$

$$\Rightarrow g'(x) = f''(x) \ \cancel{\forall x}$$
$$> 0 \quad \forall\ x.$$

$\Rightarrow g(x)$ is increasing.

Since $g(x)$ can take +ve and -ve values, so $g(x)$ can have only one root, i.e., $f'(x) = 0$ for only one x.

Let $f'(P) = 0$.

So, after P $f'(P) > 0$ because $f''(x) > 0$.

Q.141. Solution:- $f(t) = e^{-1/t}$, $t > 0$

$$\frac{d^n}{dt^n}(f(t)) = P_n(1/t)\, e^{-1/t} \;;$$

Differentiating w.r.t. t, we get $\dfrac{d^{n+1}}{dt^{n+1}} f(t)$

$$= \frac{d}{dt}\left\{P_n(1/t)\right\} \cdot e^{-1/t} + e^{-1/t}\left(\frac{1}{t^2}\right) \cdot P(1/t)$$

Now, $P_{n+1}(1/t) = e^{1/t} \cdot \dfrac{d^{n+1}}{dt^{n+1}} \cdot f(t)$

$$= e^{1/t}\left[\frac{d}{dt}\left\{P_n\left(\frac{1}{t}\right)\right\} \cdot e^{-1/t} + e^{-1/t}\left(\frac{1}{t^2}\right) P_n\left(\frac{1}{t}\right)\right]$$

$$= \frac{d}{dt}\left[P_n\left(\frac{1}{t}\right)\right] + \frac{1}{t^2} P_n\left(\frac{1}{t}\right).$$

Let us put $1/t = x$, $P_{n+1}(x) = \dfrac{d}{dx}\left\{P_n(x)\right\} \Big/ \dfrac{dt}{dx} + x^2$.

$$\therefore P_n(x) = x^2 P_n(x) - x^2 \frac{d}{dx}\left\{P_n(x)\right\}$$

$$= x^2\left(P_n(x) - \frac{d}{dx} P_n(x)\right).$$

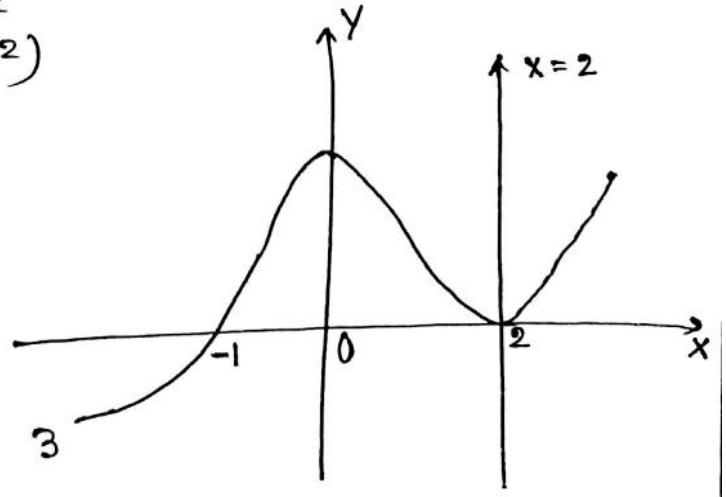
Q.142. Solution:- $f(x) = x^3 - 3x^2 + 4$

$$f'(x) = 3x^2 - 6x$$
$$= 3x(x-2)$$

$f'(x) > 0$ for $-\infty < x < 0$

$f'(x) < 0$ for $0 < x < 2$

$f'(x) > 0$ for $2 < x < \infty$



x	-2	-1	0	1	2	3
$f(x)$	-16	0	4	2	0	4

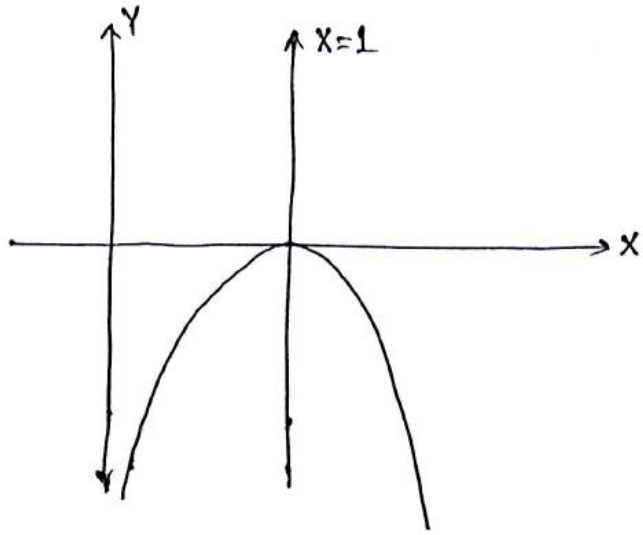
Q.143. Solution:- $f(x) = \log_e x - (x-1)$, for $x > 0$.

$$f'(x) = \frac{1}{x} - 1 = \frac{1-x}{x};$$

$f'(x) > 0$ for $0 < x < 1$

$f'(x) < 0$ for $1 < x < \infty$.

x	0.5	1	2	3
$f(x)$	0.2	0	-0.7	-1.5



Q.151. Solution:-

(a) $$x_n = \frac{1}{2}\left(x_{n-1} + \frac{5}{x_{n-1}}\right) = \frac{\sqrt{5}}{2}\left(\frac{x_{n-1}}{\sqrt{5}} + \frac{\sqrt{5}}{x_{n-1}}\right)$$

$$\Rightarrow \frac{x_n}{\sqrt{5}} = \frac{1}{2}\left(\frac{x_{n-1}}{\sqrt{5}} + \frac{\sqrt{5}}{x_{n-1}}\right).$$

By componendo-dividend, we get —

$$\frac{x_n - \sqrt{5}}{x_n + \sqrt{5}} = \frac{\frac{1}{2}\left(\frac{x_{n-1}}{\sqrt{5}} + \frac{\sqrt{5}}{x_{n-1}}\right) - 1}{\frac{1}{2}\left(\frac{x_{n-1}}{\sqrt{5}} + \frac{\sqrt{5}}{x_{n-1}}\right) + 1} = \frac{x_{n-1}^2 + \sqrt{5} - 2\sqrt{5}\,x_{n-1}}{x_{n-1}^2 + \sqrt{5} + 2\sqrt{5}\,x_{n-1}}$$

$$= \left(\frac{x_{n-1} - \sqrt{5}}{x_{n-1} + \sqrt{5}}\right)^2 = \left(\frac{x_{n-2} - \sqrt{5}}{x_{n-2} + \sqrt{5}}\right)^{2^2} = \cdots = \left(\frac{x_1 - \sqrt{5}}{x_1 + \sqrt{5}}\right)^{2^{n-1}}$$

$$= \left(\frac{x - \sqrt{5}}{x + \sqrt{5}}\right)^{2^n} \quad (\text{Proved})$$

(b) Since x is a positive number, $x - \sqrt{5} < x + \sqrt{5}$

$$\Rightarrow \frac{x - \sqrt{5}}{x + \sqrt{5}} < 1$$

$$\therefore \left(\frac{x - \sqrt{5}}{x + \sqrt{5}}\right)^{2^n} \longrightarrow 0 \text{ as } n \to \infty.$$

$$\Rightarrow \frac{x_n - \sqrt{5}}{x_n + \sqrt{5}} \longrightarrow 0 \Rightarrow x_n - \sqrt{5} \longrightarrow 0$$

$$\therefore \lim_{n \to \infty} x_n = \sqrt{5}.$$

Q.152. Solution:- $a_n = a_{n-1} + 2b_{n-1}$, $b_n = a_{n-1} + b_{n-1}$, $c_n = \frac{a_n}{b_n}$.

(a) $\sqrt{2} - c_{n-1} = \sqrt{2} - \frac{a_{n+1}}{b_{n+1}}$

$$= \sqrt{2} - \frac{a_n + 2b_n}{a_n + b_n} = \sqrt{2} - \frac{\frac{a_n}{b_n} + 2}{\frac{a_n}{b_n} + 1}$$

$$= \sqrt{2} + \frac{c_n + 2}{c_n + 1}$$

$$= \frac{\sqrt{2}c_n + \sqrt{2} - c_n - 2}{c_n + 1}$$

$$= \frac{(\sqrt{2} - 1)c_n - \sqrt{2}(\sqrt{2} - 1)}{c_n + 1}$$

$$= \frac{(\sqrt{2} - 1)(c_n - \sqrt{2})}{(c_n + 1)}.$$

(b) $$\frac{|\sqrt{2} - c_{n+1}|}{|\sqrt{2} - c_n|} = \left|\frac{1 - \sqrt{2}}{c_n + 1}\right| = \frac{1}{(1 + \sqrt{2})(c_n + 1)} < \frac{1}{1 + \sqrt{2}}$$

$$\Rightarrow |\sqrt{2} - c_{n+1}| < \frac{1}{1 + \sqrt{2}} |\sqrt{2} - c_n|$$

(c) $$\lim_{n \to \infty} \frac{c_n - \sqrt{2}}{c_{n+1} - \sqrt{2}} = \lim_{n \to \infty} \frac{c_n + 1}{1 - \sqrt{2}} = 1.$$

Q.153. Solution:-

$$\sin x_{n+1} \cos x_n - \cos x_{n+1} \sin x_{n+2} - (n+1) \sin x_n \sin x_{n+1} = 0$$

$$\Rightarrow \sin x_{n+1} \{\cos x_{n+2} - (n+1) \sin x_n\} = \cos x_{n+1} \sin x_n$$

$$\Rightarrow \cot x_{n+1} = \cot x_{n+2} - (n+1) \quad [\text{dividing by } \sin x_{n+1} \sin x_n]$$

$$\therefore \cot x_n = \cot x_{n-1} + 2^{-n} = \cot x_{n-2} + 2^{-(n-1)} + 2^{-n}$$

$$= \cot x_{n-3} + 2^{-(n-2)} + 2^{-(n-1)} + 2^{-n}$$

$$= \cdots\cdots\cdots$$

$$= \cot x_1 + 2^{-2} + 2^{-3} + \cdots + 2^{-n}$$

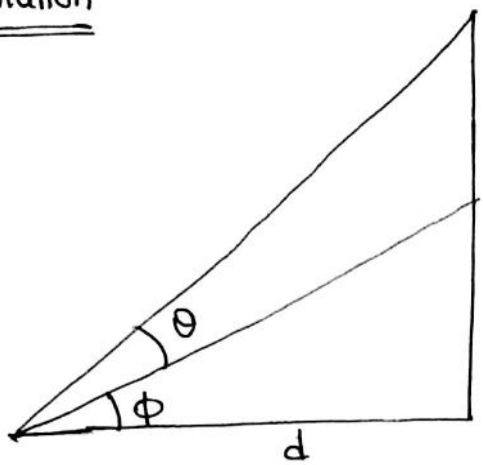
$$= 2^{-1} + 2^{-2} + 2^{-3} + \cdots + 2^{-n}$$

$$= 1 - \left(\frac{1}{2}\right)^n \quad [\because \tan^{-1} 2 = x_1]$$

$$\therefore \cot x_n = 1 - \left(\frac{1}{2}\right)^n.$$

$$\therefore \lim_{n \to \infty} \cot x_n = 1. \Rightarrow \cot\left(\lim_{n \to \infty} x_n\right) = 1 \Rightarrow \lim_{n \to \infty} x_n = \frac{\pi}{4}.$$

Q. 156. Solution



Now, when θ is maximum then length of the flagstaff is maximum.

$$\text{Length of the flagstaff} = d\left\{\tan(\theta+\phi) - \tan\phi\right\}$$

$$= d\left\{\frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} - \frac{\sin\phi}{\cos\phi}\right\}$$

$$= \frac{d\sin\theta}{\left\{\cos(\theta+\phi)\cos\theta\right\}}$$

$$= \frac{2d\sin\theta}{\left\{\cos(\theta+2\phi)+\cos\theta\right\}}.$$

Now, the expression will be maximum when denominator will be minimum, i.e., $\cos(\theta+2\phi) = 0$.

$$\therefore \quad \text{length of the flagstaff} = \frac{2d\sin\theta}{\cos\theta} = 2d\tan\theta.$$

Q.154. Solution:- $f(x) = x^2 - x\sin x$

$\therefore f(0) = 0$ and $f'(x) = 2x - \sin x - x\cos x$

$= (x - \sin x) + x(1 - \cos x)$

In the interval $[0, \pi/2]$, $x - \sin x \geq 0$ and $1 - \cos x \geq 0$.

$\therefore f'(x) \geq 0$ $\therefore f(x)$ is an increasing function of x in $[0, \frac{\pi}{2}]$.

It's min value will be $f(0) = 0$, maximum value will be $f(\pi/2) = \pi^2/4 - \pi/2$.

Q.155. Solution:-

$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots\cdots + \frac{1}{1+\frac{n}{n}}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{1}{1+\frac{r}{n}} = \int_0^1 \frac{dx}{1+x} = \left[\log_e(1+x)\right]_0^1 = \log_e 2.$$

Q.157. Solution:-

Let $P = \underset{n\to\infty}{\text{lt}} \left\{\left(1+\frac{1}{2n}\right)\left(1+\frac{3}{2n}\right)\cdots\cdots\left(1+\frac{2n-1}{2n}\right)\right\}^{1/2n}$

$$\log P = \underset{n\to\infty}{\text{lt}} \frac{1}{2n}\left\{\log\left(1+\frac{1}{2n}\right) + \log\left(1+\frac{3}{2n}\right) + \cdots + \log\left(1+\frac{2n-1}{2n}\right)\right\}$$

$$= \frac{1}{2}\underset{n\to\infty}{\text{lt}} \frac{1}{n}\sum_{r=1}^{n} \log\left(1+\frac{2r-1}{2r}\right)$$

$$= \frac{1}{2}\underset{n\to\infty}{\text{lt}} \frac{1}{n}\sum_{r=1}^{n} \log\left(1+\frac{r}{n}-\frac{1}{2n}\right)$$

$$= \frac{1}{2}\underset{n\to\infty}{\text{lt}} \frac{1}{n}\sum_{r=1}^{n} \log\left(1+\frac{r}{n}\right), \text{ since } \underset{n\to\infty}{\text{lt}} \frac{1}{2n} = 0.$$

$$= \frac{1}{2}\int_0^1 \log(1+x)\,dx$$

$$= \frac{1}{2}\left[x\log(1+x) - x + \log(1+x)\right]_0^1$$

$$= \log\left(\frac{4}{e}\right)^{1/2}$$

$$\therefore P = \left(\frac{4}{e}\right)^{1/2} = \frac{2}{\sqrt{e}}.$$

Q.159. Solution:- $\displaystyle\int_0^{\pi/2} \frac{\sin x \cos x}{(x+1)}\,dx$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sin 2x}{(x+1)}\,dx$$

$2x = z$, $2dx = dz$

x	0	$\frac{\pi}{2}$
z	0	π

$$= \frac{1}{2}\int_0^{\pi} \frac{\sin z}{\frac{z}{2}+1}\cdot\frac{dz}{2}$$

$$= \frac{1}{2}\int_0^{\pi} \frac{\sin z\,dz}{z+2} = -\frac{1}{2}\cdot\int_0^{\pi/2} \frac{\cos z}{(z+2)^2}\,dz$$

$$= \frac{1}{2}\left(\frac{1}{\pi+2}+\frac{1}{2}\right) - \frac{1}{2}A$$

$$= \frac{1}{2}\left(\frac{1}{2}+\frac{1}{\pi+2}-A\right)$$

Q.161. Solution:- $\displaystyle\int_0^{\pi}\left|\frac{\sinh x}{x}\right|dx$, as x ranges from $[0,\pi]$, so x is positive.

Let us put $nx = z$ $\therefore ndx = dz$.

$$I = \int_0^{n\pi}\left|\frac{\sin z}{z/n}\right|\frac{dz}{n} = \int_0^{n\pi}\left|\frac{\sin z}{z}\right|dz = \int_0^{\pi}\left|\frac{\sin z}{z}\right|dz + \int_{\pi}^{2\pi}\left|\frac{\sin z}{z}\right|dz + \cdots\cdots + \int_{(n-1)\pi}^{\pi}\left|\frac{\sin z}{z}\right|dz$$

Now, $\displaystyle\int_0^{\pi}\left|\frac{\sin z}{z}\right|dz = \int_0^{\pi}\frac{\sin z}{z}\,dz \geq \int_0^{\pi}\frac{\sin z}{\pi}\,dz = \frac{1}{\pi}\int_0^{\pi}\sin z\,dz$

$$= \frac{2}{\pi}.$$

Now, $\displaystyle\int_{\pi}^{2\pi}\left|\frac{\sin z}{z}\right|dz = \int_0^{\pi}\left|\frac{-\sin y}{\pi+y}\right|dy$, where $\pi+y=z$,

$$= \int_0^{\pi}\frac{\sin y}{\pi+y}\,dy \geq \frac{1}{2\pi}\int_0^{\pi}\sin y\,dy = \frac{2}{2\pi}$$

Proceeding in this way, $\displaystyle\int_{2\pi}^{3\pi}\left|\frac{\sin z}{z}\right|dz \geq \frac{2}{3\pi}, \ldots\ldots, \int_{(n-1)\pi}^{n\pi}\left|\frac{\sin z}{z}\right| \geq \frac{2}{n\pi}$

$$\therefore I \geq \frac{2}{\pi}+\frac{2}{2\pi}+\frac{2}{3\pi}+\cdots+\frac{2}{n\pi} = \frac{2}{\pi}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right).$$

Q.160. (Solution)

By induction we can solve this.

Ctanujit Classes of Mathematics,
Statistics & Economics.
Kolkata, W. B.,8420253573
website : www.ctanujit.in

For $n=0$, this is true.

Let this is true for $n=k$.

For $n=k+1$, we have

$$\int_0^{\pi/2} \frac{\sin(2k+3)x}{\sin x}\,dx = \int_0^{\pi/2} \frac{\sin\{(2k+1)x+2x\}}{\sin x}\,dx$$

$$= \int_0^{\pi/2} \left[\frac{\sin(2k+1)x\cos 2x + \cos(2k+1)x\sin 2x}{\sin x}\right]dx$$

$$= \int_0^{\pi/2} \left[\sin\{(2k+1)x\}\left(\frac{1-2\sin^2 x}{\sin x}\right)\right]dx + \int_0^{\pi/2}\left[\frac{\cos(2k+1)x \times 2\sin x\cos x}{\sin x}\right]dx$$

$$= \int_0^{\pi/2}\left\{\frac{\sin(2k+1)x}{\sin x}\right\}dx - 2\int_0^{\pi/2}\sin\{(2k+1)x\}\sin x\,dx + 2\int_0^{\pi/2}\cos\{(2k+1)x\}\cos x\,dx$$

$$= \pi/2 - 2\int_0^{\pi/2}\cos(2k+2)x\,dx$$

$$= \pi/2 - \left[2\sin(2k+2)x/(2k+2)\right]_0^{\pi/2}$$

$$= \pi/2 - 2\left[\frac{\sin(k+1)\pi - \sin 0}{(2k+2)}\right]$$

$$= \pi/2.$$

Q. 162.

We will try to show that $2\left(\sqrt{m+1}-\sqrt{m}\right) < \frac{1}{\sqrt{m}} < 2\left(\sqrt{m}-\sqrt{m-1}\right)$ —— (*)

$$2\left(\sqrt{m+1}-\sqrt{m}\right) = \frac{2\left(\sqrt{m+1}-\sqrt{m}\right)\left(\sqrt{m+1}+\sqrt{m}\right)}{\left(\sqrt{m+1}+\sqrt{m}\right)}$$

$$= \frac{2}{\left(\sqrt{m+1}+\sqrt{m}\right)} < \frac{2}{2\sqrt{m}} = \frac{1}{\sqrt{m}}.$$

Similarly, we can show $2\left(\sqrt{m}-\sqrt{m-1}\right) > \frac{1}{\sqrt{m}}$.
in (*)

Now, putting $m = 1, 2, \ldots, 250$ and adding the inequalities we will get

$$2\left(\sqrt{251}-1\right) < \sum_{K=1}^{250} \frac{1}{\sqrt{K}} < 2\left(\sqrt{250}\right).$$

Q. 163.

$$\log x = \int_1^x \frac{dt}{t}, \quad x > 0.$$

$$\log\left(1+\frac{1}{n}\right) = \int_1^{1+\frac{1}{n}} \frac{dt}{t}$$

Now, $t \le 1+\frac{1}{n} \Rightarrow \frac{1}{t} \geq \frac{n}{n+1}$.

$$\therefore \int_1^{1+\frac{1}{n}} \frac{dt}{t} \geq \frac{n}{n+1}\int_1^{1+\frac{1}{n}} dt$$

$$\Rightarrow \log\left(1+\frac{1}{n}\right) \geq \frac{1}{n+1}.$$

Again, $t \geq 1$, $\frac{1}{t} \le 1$.

$$\therefore \int_1^{1+\frac{1}{n}} \frac{dt}{t} \le \int_1^{1+\frac{1}{n}} dt = \frac{1}{n}.$$

$\Rightarrow \log\left(1+\frac{1}{n}\right) \le \frac{1}{n}$. So, we have $\frac{1}{n+1} \le \log\left(1+\frac{1}{n}\right) \le \frac{1}{n}$.

Q.164. Solution:- $y^3 = x^2$ .....(i) $\qquad y = 2 - x^2$ ..... (ii)

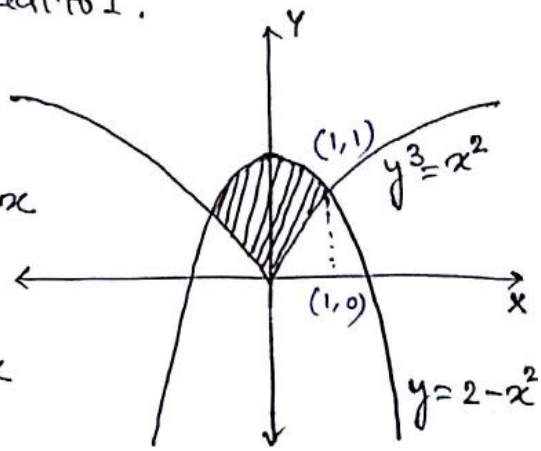
$\Rightarrow y = 2 - y^3$

$\Rightarrow (y-1)(y^2+y+2) = 0$

$\therefore y = 1, \; y = \frac{1}{2}(-1 \pm \sqrt{3})$

$\therefore x = \pm 1$, since y is real & equal to 1.

Point of intersection of the two curves are $(1,1)$ and $(-1,1)$.



$\therefore$ Area of shaded region $= \int_{-1}^{1} (y_1 - y_2)\, dx$

$$= \int_{-1}^{1} \left[2 - x^2 - x^{2/3} \right] dx$$

$$= 2\frac{2}{15} \text{ sq. units.}$$

Q.165. Solution:- $y = x^2$ .....(i) $\qquad x + y = 2$ ..... (ii)

$y = -\sqrt{x}$ .....(iii)

So, $x^4 = x$, or, $x(x^3 - 1) = 0$, so, $x = 0, 1$, from (i) & (iii)

And, $(2-x)^2 = x$, or, $x^2 - 5x + 4 = 0$ or, $x = 1, 4$, from (ii) & (iii)

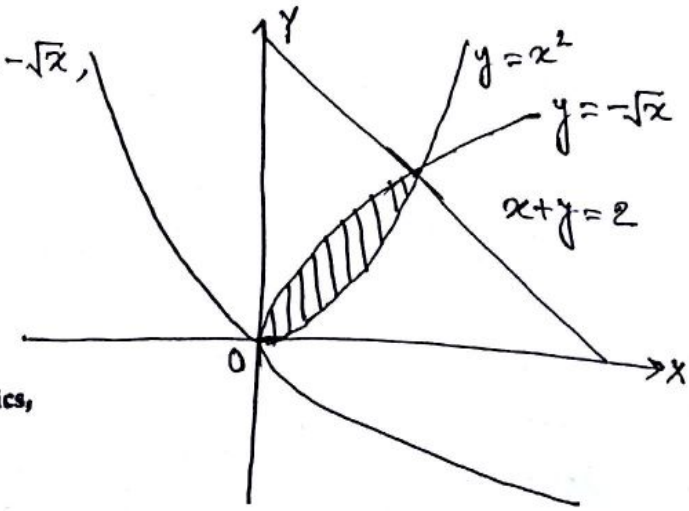
So, Point of intersection is $x = 1$.

Area of the shaded region is

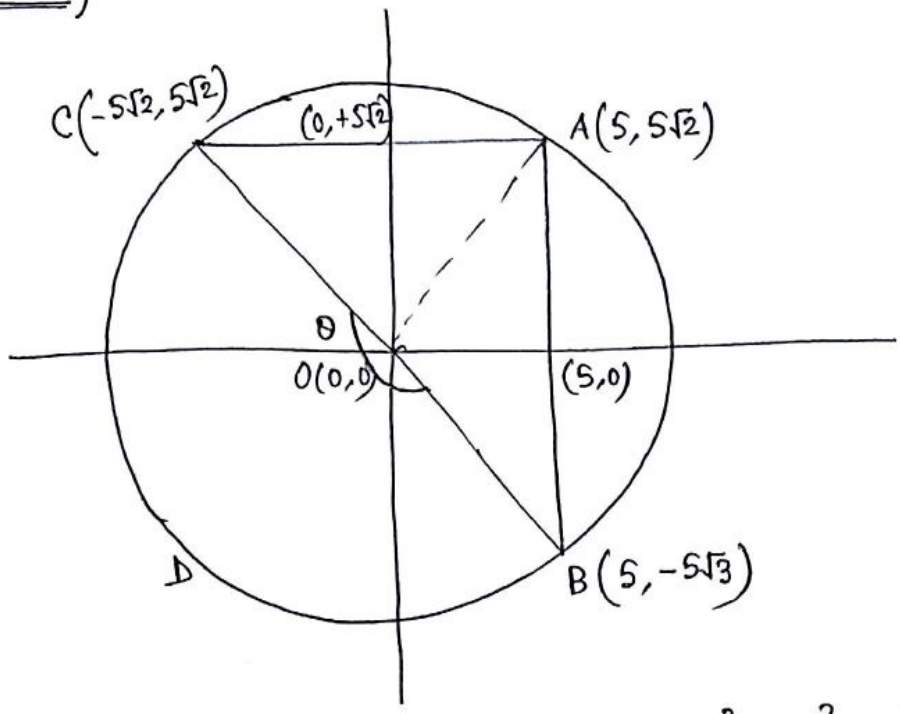
$= \left| \int_0^1 (y_2 - y_1)\, dx \right|$, where $y_1 = -\sqrt{x}$, $y_2 = x^2$

$= \left| \int_0^1 (-\sqrt{x} - x^2)\, dx \right|$

$= 1$ sq. units.



Q.166. (Solution)



By solving the equation of the circle $x^2+y^2=10^2$ and the straight lines $x=5$ and $y=5\sqrt{2}$ and we get the co-ordinate of the points A, B, C as given in the given figure.

Now, slope of line OB $= -\frac{5\sqrt{3}}{5} = -\sqrt{3}$ and slope of the line OC $= \frac{5\sqrt{2}}{-5\sqrt{2}} = -1$. (negative sign refers negative axis)

So, $\tan\theta = \frac{(-1+\sqrt{3})}{(1+\sqrt{3})} = 2-\sqrt{3}$.

$\Rightarrow \theta = \tan^{-1}(2-\sqrt{3})$

Area of OCDB $= \frac{\pi \times 10^2}{2\pi} \times \tan^{-1}(2-\sqrt{3}) = 50\tan^{-1}(2-\sqrt{3})$.

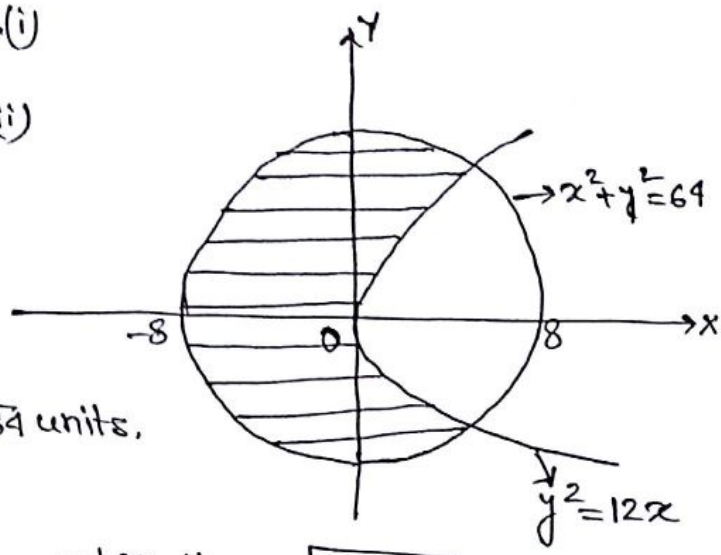
Area of triangle OAB $= \frac{1}{2} \times 5 \times (5\sqrt{2}+5\sqrt{3}) = \frac{25}{2}(\sqrt{3}+\sqrt{2})$.

Area of triangle OAC $= \frac{1}{2} \times 5\sqrt{2} \times (5+5\sqrt{2}) = \frac{25}{2}(\sqrt{2}+1)$.

The cow can graze in the area = area of OCDB + area OAB + area OAC.

Q.167. Solution:- $x^2+y^2=64$ .....(i)

$y^2=12x$ .....(ii)



$\Rightarrow 64-x^2=12x$

$\Rightarrow (x-4)(x+16)=0$

$\therefore x=-16, 4$

but $x=-16$ is not possible as radius of a circle is $\sqrt{64}$ units.

Area of the shaded region is

$$=2\int_0^4 (y_1-y_2)\,dx+\frac{64\pi}{2}, \text{ where } y_1=\sqrt{64-x^2},\ y_2=2\sqrt{3x}$$

$$=2\int_0^4\left(\sqrt{64-x^2}-2\sqrt{3x}\right)dx+32\pi$$

$$=2\left[\frac{x\sqrt{64-x^2}}{2}+\frac{64}{2}\sin^{-1}\frac{x}{8}-2\sqrt{3}\cdot\frac{x^{3/2}}{3/2}\right]_0^4+32\pi$$

$$=\frac{16}{3}\left(8\pi-\sqrt{3}\right) \text{ sq. units.}$$

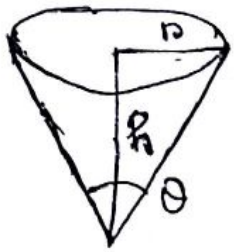
Q.168. Solution:-

Perimeter of the marked region of the circle $=2\pi a-a(2\pi-\theta)$

$=a\theta$

Perimeter of the base of the cone $=2\pi r$

$\therefore 2\pi r=a\theta \Rightarrow r=\frac{a\theta}{2\pi}$



Volume of the cone $=V=\frac{1}{3}\pi r^2 h$

$$=\frac{1}{3}\pi\cdot\frac{a^2\theta^2}{4\pi^2}\sqrt{a^2-\frac{a^2\theta^2}{4\pi^2}}$$

$$=\frac{a^3}{24\pi^2}\theta^2\sqrt{4\pi^2-\theta^2}$$

$$\therefore \frac{dV}{d\theta}=\frac{a^3}{24\pi^2}\left(2\theta\sqrt{4\pi^2-\theta^2}-\frac{\theta^3}{\sqrt{4\pi^2-\theta^2}}\right)$$

for V to be max or min, $\frac{dV}{d\theta}=0$

$$\therefore 2\theta\sqrt{4\pi^2-\theta^2} - \frac{\theta^3}{\sqrt{4\pi^2-\theta^2}} = 0$$

$$\Rightarrow 8\pi^2 - 2\theta^2 - \theta^2 = 0$$

$(\because \theta \neq 0)$

$$\therefore \theta = 2\pi\sqrt{\frac{2}{3}}$$

$$\left[\frac{d^2v}{d\theta^2}\right]_{\theta=2\pi\sqrt{\frac{2}{3}}} = \frac{a^3}{24\pi^2}\left[2\sqrt{4\pi^2-\theta^2} - \frac{1}{2}\frac{4\theta}{\sqrt{4\pi^2-\theta^2}} - \frac{3\theta^2}{\sqrt{4\pi^2-\theta^2}} + \frac{2\theta^4}{(4\pi^2-\theta^2)^{3/2}}\right]_{\theta=2\pi\sqrt{\frac{2}{3}}} < 0$$

$\therefore$ The volume of the funnel is max. when θ equals $2\pi\sqrt{\frac{2}{3}}$

[proved]

Q. 169. Solution:-

Suppose, O be the centre of the circle which lies in the star and where as $\angle MOL = \theta$, so $\angle LAM = \frac{\theta}{2}$.

Here, $5 \cdot R \cdot \theta = 2\pi R$

$$\therefore \quad \theta = \frac{2\pi}{5}.$$

from ΔOAL,

$\angle AOL = 2\frac{\pi}{5} \Rightarrow \frac{1}{2} = \frac{\pi}{5}$.

$\angle LAO = \frac{1}{2}\angle AOL = \frac{\pi}{10}$

$\angle OLA = \pi - (\angle AOL + \angle LAO)$

$= \pi - \left(\frac{\pi}{5} + \frac{\pi}{10}\right)$

$= \frac{7\pi}{10}$

Let, $AL = a$, $OL = b$ & $OA = R$ (given)

$\therefore$ From ΔOAL, we have, $\dfrac{a}{\sin \pi/5} = \dfrac{b}{\sin \pi/10} = \dfrac{R}{\sin 7\pi/10}$

$\therefore a = R\dfrac{\sin \pi/5}{\sin 7\pi/10}$ & $b = R\dfrac{\sin \pi/10}{\sin 7\pi/10}$

Thus area of $\Delta AOL = \frac{1}{2}ab \sin\frac{7\pi}{10} = \frac{1}{2}R^2$

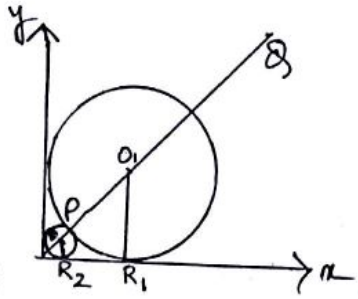
$$\frac{\sin\frac{\pi}{5}\sin\frac{\pi}{10}}{\sin^2 7\frac{\pi}{10}} \times \sin 7\frac{\pi}{10} = \frac{1}{2}R^2\, 2\sin\frac{\pi}{10}\cos\frac{\pi}{10}\sin\frac{\pi}{10}$$

$$\left[\because \sin 7\frac{\pi}{10} = \sin 3\frac{\pi}{10}\right] = \frac{R^2\sin^2\frac{\pi}{10}\cos\frac{\pi}{10}}{3\sin\frac{\pi}{10} - 4\sin\frac{3\pi}{10}} = \frac{R^2\tan\frac{\pi}{10}}{3-\tan^2\frac{\pi}{10}}$$

[Dividing N^{r} & D^{r} by $\sin\frac{\pi}{10}\cos^2\frac{\pi}{10}$]

Hence, required area of the star $= \dfrac{10R^2\tan\frac{\pi}{10}}{3-\tan^2\frac{\pi}{10}}$ [Proved]

Q. 170. Solution:-



$OO_1 = R_1\sqrt{2}$ $\therefore OP = R_1\sqrt{2} - R_1$

$\therefore OQ = R_1\sqrt{2} + R_1 = R_1(\sqrt{2}+1)$

$\therefore R_1 = \dfrac{OQ}{\sqrt{2}+1}$, Now, $OP = R_2(\sqrt{2}+1)$, $R_2 = \dfrac{OP}{\sqrt{2}+1} = R_1\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$

$\therefore R_3 = R_2\dfrac{\sqrt{2}-1}{\sqrt{2}+1} = R_1\left(\dfrac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2$ $\therefore$ Area $= \pi(R_1^2 + R_2^2 + \cdots \infty)$

$$= \pi\left\{R_1^2 + R_1^2\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2 + R_1^2\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^4 + \cdots \infty\right\}$$

$$= \pi R_1^2\left\{1 + \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2 + \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^4 + \cdots \infty\right\}$$

$$= \pi R_1^2\left\{\frac{1}{1-\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2}\right\} = \pi R_1^2\,\frac{3+2\sqrt{2}}{4\sqrt{2}} = \pi R_1^2\,\frac{3\sqrt{2}+4}{8}$$

$$= \frac{\pi}{8}R_1^2\,\frac{18-16}{3\sqrt{2}-4} = \frac{\pi}{4}\cdot 100\cdot\frac{1}{3\sqrt{2}-4}\ \text{sq. cm}\quad [\because R_1 = 10\text{cm}]$$

$$= \frac{25\pi}{3\sqrt{2}-4}\ \text{sq. cm}$$ [proved]

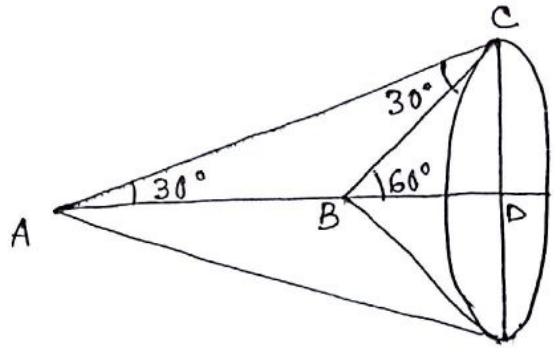
Q. 171. Solution:-

Here $AB = BC = 1$

$BD = BC \cos 60° = 1 \cdot \frac{1}{2} = \frac{1}{2}$

$CD = BC \sin 60° = 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

$\therefore AD = AB + BD = 1 + \frac{1}{2} = \frac{3}{2}$



$\therefore$ Required volume

$= \frac{1}{3} \pi CD^2 (AD - BD)$

$= \frac{1}{3} \pi \cdot \frac{3}{4}$

$\therefore \left(\frac{3}{2} - \frac{1}{2}\right)$ sq. unit $= \frac{1}{3} \cdot \frac{3\pi}{4}$ sq. unit

$= \frac{\pi}{4}$ sq. unit.

Q. 172. Solution:-

By the problem, every team will play $K-1$ matches. As the i-th team loses l_i & wins w_i matches, they play a total of $l_i + w_i$ matches, as no match ends in draw.

$\therefore l_i + w_i = K-1 \quad \ldots (1)$

Obviously, total no. of wins in the tournament

= total no. of loses in the tournament

$\Rightarrow \sum_{i=1}^{K} l_i = \sum_{i=1}^{K} w_i \quad \ldots (2)$

Now, $\sum_{i=1}^{K} l_i^2 - \sum_{i=1}^{K} w_i^2 = \sum_{i=1}^{K} (l_i^2 - w_i^2)$

$= \sum_{i=1}^{K} (l_i + w_i)(l_i - w_i) = \sum_{i=1}^{K} (K-1)(l_i - w_i)$ [from (1)]

$= (K-1)\left\{\sum_{i=1}^{K} l_i - \sum_{i=1}^{K} w_i\right\} = (K-1) \cdot 0$ [from (2)]

$= 0$

$\Rightarrow \sum_{i=1}^{K} l_i^2 = \sum_{i=1}^{K} w_i^2$ (proved)

Q. 173. Solution-

According to the question, $P_1 = P_1^2 + P_2^2 + P_3^2 + \dots + P_n^2$

This is possible when the degree of P_1 is 0.

$P_1^2 \geq P_1$, $\underline{P_2^2 + P_3^2 + \dots + P_n^2 \geq 0}$ Given that $P_1^2 + P_2^2 + P_3^2 + \dots$

$P_1^2 + P_2^2 + \dots + P_n^2 \geq$

$\dots + P_n^2 = P_1$, $P_1^2 = P_1$, $\because P_1 \neq 0$ $\therefore P_1 = 1$.

And $P_2^2 + P_3^2 + \dots + P_n^2 = 0$

i.e., $P_2 = P_3 = \dots P_n = 0$ [proved]

Q. 174. Solution:-

$P(x) = x^4 + ax^3 + bx^2 + cx + d$, Now, sum of the roots $= -a$

$\Rightarrow \alpha + \beta + \gamma + \delta = -a$

again, $\alpha + \beta = +1$, $\gamma + \delta = 10$, $\alpha + \gamma = 2$, $\alpha + \delta = 6$, $\beta + \gamma = 5$,

$\beta + \delta = 9$, Adding, $3(\alpha + \beta + \gamma + \delta) = 33$

$\Rightarrow \alpha + \beta + \gamma + \delta = 11$

$\Rightarrow a = -11$

Solving, the equations,

$\alpha = -1$ [Note: $(\alpha + \gamma)$ and $(\alpha + \delta)$ both should be either

$\beta = 2$ even or both should be odd, else a, b, c, d

$\gamma = 3$ will not be integers]

$\delta = 7$

We know, $\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \beta\delta = b$

$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \alpha\beta\delta = -c$

$\alpha\beta\gamma\delta = d$

putting the values of $\alpha, \beta, \gamma, \delta$, $b = 29$, $c = -1$, $d = -42$, $a = -11$.

$\therefore$ a, b, c, d any integers.

$\therefore P(x) = x^4 - 4x^3 + 29x^2 - x - 42$ and, $P(1/2)_2 = \frac{585}{16}$

Q.175. Solution:-

$P(0) = a_0 = \text{odd}$, $P(1) = 1 + a_{n-1} + a_{n-2} + \cdots + a_0 = \text{odd}$

(a) Case-I If $x = 2m$, then

$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 = \text{odd}$, as all the term containing x will be even but $a_0 = \text{odd}$.

$\Rightarrow x = 2m$ cannot be a root of the eqn $P(x) = 0$ [proved]

(b) Case-II: If $x = 2m+1$

$P_x = \underbrace{x^n}_{\text{odd}} + \underbrace{a_{n-1}x^{n-1}}_{\text{odd}} + \cdots + \underbrace{a_0}_{\text{odd}} = x^n + a_{n-1}(\text{even}+1) + a_{n-2}(\text{even}+1)$

$+ \cdots + a_1(\text{even}+1)a_0 = x^n + a_0 + (a_{n-1}\,\text{even} + a_{n-2}\,\text{even} + \cdots$

$\cdots + a_1\,\text{even}) + (a_1 + b_2 + \cdots + a_{n-1}) = \underbrace{x^n}_{\text{odd}} + (a_{n-1}\,\text{even}$

$+ \underbrace{a_{n-2}\,\text{even}}_{\text{even}} + \cdots + a_1\,\text{even}) + (a_0 + a_1 + \cdots + a_{n-1})$

$= \text{odd} \Rightarrow x = (2m+1)$ cannot be a root of the eqn $P(x) = 0$ [proved].

Q.176. Solution:-

Given that, $P(x)$ is a polynomial of degree n such that

$P(K) = \frac{K}{K+1} \quad \forall\ K = 0, 1, \ldots, n$

let, $Q(x) = (x+1)P(x) - x$

The polynomial $Q(x)$ vanishes for $K = 0, 1, \ldots, n$

i.e., $(x+1)P(x) - x = a(x)(x-1)(x-2)\cdots(x-n)$

putting $x = -1$, $1 = a(-1)(-2)(-3)\cdots(-1-n)$

$\Rightarrow 1 = a(-1)^{n+1}(n+1)! \Rightarrow a = \frac{1}{(-1)^{n+1}} \cdot \frac{1}{(n+1)!}$

$$P(x) = \frac{ax(x-1)(x-2)\cdots(x-n)+x}{(x+1)}$$

$$= \frac{(-1)^{n+1}x(x-1)(x-2)\cdots(x-n)}{(n+1)!} + x$$

$$\therefore P(n+1) = (-1)^{n+1}\frac{\{(n+1)n(n-1)\cdots 3.2.1\}+x+1}{(n+1)!} \cdot \frac{1}{(n+2)}$$

$$= (-1)^{n+1}\frac{\frac{(n+1)}{(n+1)!}+(n+1)}{(n+2)} = \frac{(-1)^{n+1}+n+1}{(n+2)}$$

$$\therefore P(n+1) = \begin{cases} \frac{1}{n} & \text{for } n = \text{odd} \\ n+2 & \text{for } n = \text{even} \end{cases}$$

Q. 177. Solution:-

By the problem, the person P_m will change the state of the door D_n, where $m \mid n$ i.e., m is one of the factors of n.

At first, all the doors were closed and we are to determine the no. of doors remaining open, i.e., we are to determine the no. of doors whose states are finally changed.

Now, for the door D_n, n will either have even or odd no. of factors.

It is obvious, for even no. of factors, state of doors will remain same, so whenever no of fact $\sigma(n)$ of n is odd, state of D_n changes.

$\Rightarrow \sigma(n) = 2K+1, \quad K \in I^+ \Rightarrow \sigma(n) = (2p+1)(2q+1) \cdots$
$\cdots (2m+1) \cdots ; \Rightarrow n = a^{2p} b^{2q} c^{2r} \ldots$ for prime $a, b, c \ldots$

$\Rightarrow$ Square numbered door remains open.

$\therefore$ No. of doors remaining open $= (\sqrt{1000}) = 31$

Q. 179. Solution :

$x * y = 1, \Rightarrow xy = 100K+1$ for $x = \{0, 1, 2, \ldots, 99\}$

(1) For $x = 1, y = 1, x * y = 100K + 1$ where $x = c$

$\therefore$ Inverse of 1 is 1.

(2) There is no integral multiple of 2, 4, 5, 6 having 1 at unit place, $\Rightarrow$ 2, 4, 5, 6 have no inverse.

(3) 3 and 7 can have inverses.

(i) For $x = 3$, $3y = 1$ i.e., $3y = 100K + 1$

The least K satisfying is 2, i.e., $3y = 201$, $y = 67$ and the next K satisfying is 5, i.e., $3y = 501$, $y = 167$ but $167 \notin X$

$\therefore$ 3 has only inverse $= 67$.

(ii) for $x = 7$, $7y = 1$, i.e., $7y = 100K + 1$

The least K satisfying is 3, i.e., $7y = 301$, $y = 43$

The next K satisfying is 10, i.e., $7y = 1001$, $y = 143$ but $143 \notin X$.

$\therefore$ 7 has only inverse $= 43$.

Q.178. Now, to cross b columns it must have gone through b unit squares.

Now, to go l rows down, it has to go through l−1 number of unit squares as to go from 1st row to second row it has to go to one additional unit square, to go from second row to third row it has to go to one additional unit square.

So, number of unit squares contains a segment of positive length of this diagonal = b+l−1.

Now, the bottom right corner has coordinate (b−1, 0) and top left corner have coordinate (0, l−1).

The equation of the diagonal is, $\frac{x}{b-1} + \frac{y}{l-1} = 1$.

Note that, this straight line passes through $\left\{\frac{(b-1)}{d}, (l-1) - \frac{(l-1)}{d}\right\}$

Now this point will be integer if d divides both (l−1) and (b−1).

Another point passing through straight line $\left\{(b-1) - \frac{(b-1)}{d}, \frac{(l-1)}{d}\right\}$

So, total number of unit squares = l+b−1−2m, where, m is the number of common divisors of l−1 and b−1.

Now, again note that if d=2 then $\frac{(b-1)}{d} = (b-1) - \frac{(b-1)}{d}$

So, two points will be counted as one.

So, if (b−1) and (l−1) are both even, i.e., if they have a common factor 2 then total number of unit squares = l+b−1−2m+1 = l+b−2m.

Q.180. Solution:-

Kin \ Friend	Yes	No.	Total
Yes	27	31	58
No.	3	129	132
Total	30	160	190

1. Most of the friends are Kin because 3 of 30 friends are not kin.
2. Most of the friends are kin, which is not true.

Q.182. Solution:-

	A	B	C	D	E	F	Final score
A	X	0	2	0	2	2	6
B	2	X	2	2	0	2	8
C	0	0	X	$\boxed{2}$	$\boxed{2}$	0	4
D	2	0	$\boxed{0}$	X	2	2	6
E	0	2	$\boxed{0}$	0	X	2	4
F	0	0	2	0	0	X	2

Steps:

(1) First use (i) then (iii)

(2) Since after 4 games, B had 8 pts. and B lost to E later, so B had won against A, C, D, F.

(3) Since A had 6 pts after 4 games and A had lost to B and D, so A won against C, E and F.

(4) C had 4 pts. after 4 games. So, C won against D and E.

(5) Since, A, B and D moved to final round and C has

total 4 pts, so D must have won the games against E and F.

Q. 187. Solution:-

Let us try to write,

$$x^3 + x + 1 = (ax+b)(cx^2+dx+e)$$

$$= acx^3 + (bc+ad)x^2 + (bd+ae)x + be;$$

$\Rightarrow ac = 1, \quad bc + ad = 0, \quad bd + ae = 1, \quad be = 1.$

From the given rule, $a = 1, \ c = 1, \ b = 1, \ e = 1.$

$\therefore \quad bc + ad = 0$, substituting the rules,

$1 + d = 0$

$\Rightarrow d = 1 \quad (\because 1 + 1 = 0)$

Again, $bd + ae = 1$, substituting the rules,

$1 + 1 = 1$, but, by the rule $1 + 1 = 0$,

which contradicts our assumption.

$\Rightarrow x^3 + x + 1$ can't be factorial in this system. [Proved]

Q.185.

Clearly, $(A \cup B) \setminus (A \cap B) = (A \cap B^c) \cup (A^c \cap B)$

Now, $(A * B) * C = [(A \cap B^c) \cup (A^c \cap B) \cap C^c] \cup [\{(A \cap B^c) \cup (A^c \cap B)\}^c \cap C]$

$= (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A \cap B \cap C)$

[Use Venn diagram to show]

Similarly, we can show

$A * (B * C) = (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A \cap B \cap C)$

[Use Venn diagram to show]

$= (A * B) * C$

Now, $e = \phi$ (the null set) and $A' = A$.

Q.188.

We can write the numbers in first row as

$0 \times 8 + 1, 0 \times 8 + 2, \ldots\ldots, 0 \times 8 + 8$.

We can write the numbers in second row as

$1 \times 8 + 1, 1 \times 8 + 2, \ldots\ldots, 1 \times 8 + 8$.

We can write the numbers in third row as

$2 \times 8 + 1, 2 \times 8 + 2, \ldots\ldots, 2 \times 8 + 8$.

Similarly, for other rows.

So, if we choose 8 numbers from different rows, then the numbers will be $0 \times 8 + a_1 + 1 \times 8 + a_2, \ldots, 7 \times 8 + a_8$; where, $a_1, a_2, \ldots, a_8$ are all less than or equal to 8 and distinct as they are from different columns.

So, we have, $a_1 + a_2 + \cdots + a_8 = 1 + 2 + \cdots + 8 = 36$.

Now, the sum of the 8 chosen numbers $= 0 \times 8 + a_1 + 1 \times 8 + a_2 + \cdots + 7 \times 8 + a_8$

$$= 8(1 + 2 + \cdots + 7) + (a_1 + a_2 + \cdots + a_8)$$

$$= \frac{8 \times 7 \times 8}{2} + 36$$

$$= 224 + 36$$

$$= 260.$$

Ctanujit Classes of Mathematics, Statistics & Economics.
Kolkata, W. B.,8420253573
website : www.ctanujit.in